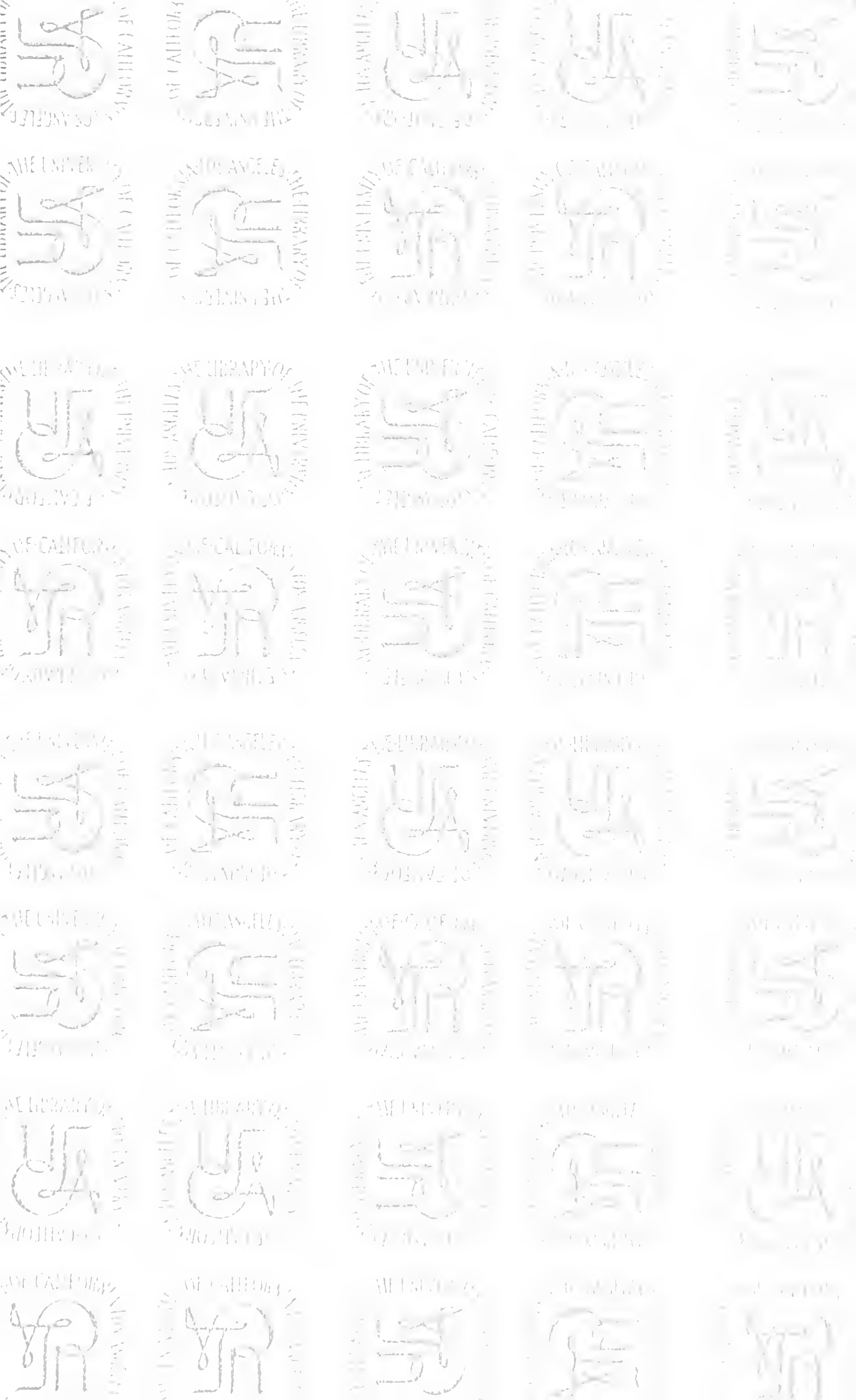

THE ADJUSTMENT

OF

OBSERVATIONS

BY THE METHOD OF LEAST SQUARES WITH APPLICATIONS TO GEODETIC WORK

BY

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PREFACE

THIS book originated in this way. While employed as Assistant Engineer on the Survey of the Northern and Northwestern Lakes, many questions came up in the course of the work for which no help could be found in any publication in the library of the Survey. Conclusions were, in general, reached often after long continued discussions. I at the time made notes of the questions and of the solutions obtained, in order that if similiar questions should again come up they might more readily be dealt with.

At the close of the Survey, I had a large collection of notes of this kind. Shortly afterwards, on entering college work, these notes were arranged in systematic order. Also at the same time an account of the Coast and Geodetic Survey methods of work was added.

As only a comparatively small edition was printed from type in the first place, the book has been out of print for a number of years, though repeated requests have been made for copies. In the spring of 1903, Superintendent Tittmann, U. S. Coast and Geodetic Survey, wrote to the publishers as follows: "As this book is one of exceeding importance to the Survey, and will grow even more needful in this work, and, as I take it, in many fields of scientific engineering operations, I beg to inquire whether you anticipate issuing a new edition of this useful book?"

This led to some correspondence, and it was finally arranged that Mr. John F. Hayford, Chief of the Computing Division and Inspector of Geodetic Work, should assist in revising the book. Most of the new matter has been contributed by Mr.

Hayford. Indeed, so important have been his contributions that I have insisted that his name appear on the title page.

We have been assisted by the leading members of the Computing Division, notably by Mr. M. H. Doolittle. For Chapters VII and IX, Mr. Hayford is alone responsible. Chapter VII contains an authoritative account of the latest methods in use in the Coast Survey for the adjustment of a triangulation. The account given in the first edition was, at the time of its publication, nearly correct. But during the last twenty years rapid progress has been made in the Computing Division of the C. & G. S. in improving methods of computation, under the direction of Mr. C. A. Schott and his successor Mr. Hayford, with the able assistance of Mr. Doolittle, Mr. E. H. Courtenay, Mr. C. H. Kummel, Mr. A. L. Baldwin, and others, who have done original thinking. The changes are the result of seven decades of experience in the Survey.

Chapter IX contains an exceedingly important application of the method of least-squares to the selection of methods of observation.

The leading principles that have been followed in making the revision are these:

1. Matter that was curious only and without application has been omitted.

2. Matter relating to description of instruments and methods of observation is in general eliminated.

3. Statements of formulas not pertaining to least-squares are omitted.

4. Following American custom, the term "probable error" is used instead of "mean-square error."

5. In order not to increase the size of the book all applications to Physics, etc., have been omitted.

The number of pages has been cut down from four hundred and thirty-seven to 303.

T. W. W.

SCHENECTADY, Nov., 1905.

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THE ADJUSTMENT OF OBSERVATIONS

CHAPTER I

INTRODUCTION

The factors that enter into the measurement of a quantity are, the observer, the instrument employed, and the conditions under which the measurement is made.

1. **The Instrument.** — If the measure of a quantity is determined by untrained estimation only, the result is of little value. The many external influences at work hinder the judgment from deciding correctly. For example, if we compare the descriptions of the path of a meteor as given by a number of people who saw the meteor and who try to tell what they saw, it would be found impossible to locate the path satisfactorily. The work of the earlier astronomers was of this vague kind. There was no way of testing assertions, and theories were consequently plentiful.

The first great advance in the science of observation was in the introduction of instruments to aid the senses. The instrument confined the attention of the observer to the point at issue, and helped the judgment in arriving at conclusions. As with a rude instrument different observers would get the same result, it is not to be wondered at that for a long time a single instrumental determination was considered sufficient to give the value of the quantity measured.

The next advance was in the questioning of the instrument and in showing that a result better on the whole than a single direct measurement could be found. This opened the way for better instruments and better methods of observation. For example, Gascoigne's introduction of cross-hairs into the focus of

the telescope led to better graduated circles and to better methods of reading them, resulting finally in the reading microscopes now almost universally used. The culminating point was reached by Bessel, who, by his systematic and thorough investigation of instrumental corrections and methods of observation, may be said to have almost exhausted the subject. He confined himself, it is true, to astronomical and geodetic instruments, but his methods are of universal application.

The questioning of an instrument naturally arises from noticing that there are discrepancies in repeated measurements of a magnitude with the same instrument, or in measures made with different instruments. Thus, if a distance was measured with an ordinary chain, and then measured with a standard whose length had been very carefully determined, and the two measurements differed widely, we should suspect the chain to be in error, and proceed to examine it before further measuring. So, discrepancies found in measurements made with the same measure at different temperatures have shown the necessity of finding the length of the measure at some fixed temperature, and then applying a correction for the length at the temperature at which the measurement is made.

Corrections to directly measured values are thus seen to be necessary, and to be due to both internal and external causes. The internal causes arising from the construction of the instrument are seen to be in great measure capable of elimination. From geometrical considerations the observer can tell the arrangement of parts demanded by a perfect instrument. He can compute the errors that would be introduced by certain supposed irregularities in form and changes of condition. The instrument-maker cannot, it is true, fulfill the conditions necessary for a perfect instrument, but he has been gradually approaching them more and more closely. It is to be remembered that, even if an instrument could be made perfect at any instant, it would not remain so for any great length of time.

It hence followed as the next great advance that the instru-

ment was made adjustable in most of its parts, so that the relative positions of the parts are under the control of the observer. This is getting to be more and more the case with the better class of instruments.

2. Not only is error diminished by the improved construction of the instrument, but also by more refined methods of handling it. It may be, indeed, that some contrivances beyond those required to make necessary readings for the measure of the quantity in question may be needed. Thus, with a graduated circle, regular or periodic errors of graduation may be expected. If the angle between two signals were read with a theodolite, the reading on each signal, and consequent value of the angle, would be influenced by the periodic errors of the circle of the instrument. Though a single vernier or microscope would suffice to read the circle when the telescope is directed to the signals, yet, as the circle is incapable of adjustment, we can only get rid of the influence of the periodicity by employing a number of verniers or microscopes placed at equal intervals around the circle. It happens that this same addition of microscopes eliminates eccentricity of the graduated circle as well.

This same principle of making the method of observation eliminate the instrumental errors is carried through even after the nicest adjustments have been made. Thus, in ordinary leveling, if the backsights and foresights are taken exactly equal the instrumental adjustment may be poor and still good work may be done. But good work is more likely if the adjustments have been carefully made, as if for unequal sights, and still the sights are taken equal.

Simplicity of construction in an instrument is also to be aimed at. An instrument that theoretically *ought* to work perfectly is often a great disappointment in practice. A striking example is the compensating base apparatus which has been abandoned on all the leading surveys. The mechanical and thermal difficulties have proved to be insurmountable, and the compensating bars have been replaced by others of much simpler construction.

Similarly, the repeating theodolite has fallen far short of the expectations of its first advocates, who hoped that with it the errors of measurement of an angle could be reduced almost indefinitely. The mechanical difficulties have proved insurmountable, and the repeating theodolite is now known to be capable of no greater accuracy than the direction instrument.

Such is the perfection at present attained in the construction of mathematical instruments, and the skill with which they can be manipulated, that comparatively little trouble in making observations arises from the instrument itself.

3. External Conditions. — The great obstacles to accurate work arise from the influence of external conditions — conditions wholly beyond the observer's or instrument-maker's control, and whose effect can, in general, neither be satisfactorily computed nor certainly eliminated by the method of observation. We have no means of finding the complex laws of their action. Many of them can be *avoided* by not observing while they operate in any marked degree. Thus, if while an observer is reading horizontal angles on a high tower a strong wind arises, it may be necessary for him to stop work. If the air commences to "boil" with extreme violence, it may also be best to stop work. If the sun shines on one side of his instrument, its adjustments would be so disturbed that good work could not be expected. So in comparisons of standards. Comparisons made in a room subject to the temperature variations of the outside air would be of little value. The standards should not only not be exposed to sudden temperature changes during comparisons, but at no other time; for it has been shown that the same standard may have different lengths at the same temperature after exposure to wide ranges of temperature.

The effects of external disturbances may sometimes be eliminated, in part at least, by the method of observation. In the measurement of horizontal angles where the instrument is placed on a high tower, the influence of the sun causes the

center post or tripod of the station to twist in one direction during the day. When this influence is removed at night, the twist is in the opposite direction. Assuming the twist to act uniformly, its effect on the results is eliminated by taking the mean of the readings on the signals observed in order of azimuth, and then immediately in the reverse order.

Atmospheric refraction is another case in point. In observing for time with a sextant, the effect of refraction is often eliminated when the highest degree of accuracy is required, by taking two sets of observations of the sun at about the same altitude, one before and the other after noon. On the other hand, in the measurement of horizontal angles, if long lines are sighted over, or lines passing from land over large bodies of water, or over a country much broken, the effects of refraction are apt to be very marked. As we have no means of eliminating the discordances arising in this way by the method of observation, all we can do is, while planning a triangulation, to avoid as far as possible the introduction of such lines.

It may happen that the effect of the external disturbances on the observations can be computed approximately from theoretical considerations assuming a certain law of operation. If the correction itself is small, this is allowable. As an example, take the zenith telescope, with which the method of observing for latitude is such that the correction for refraction is so small that the error of the computed value is not likely to exceed other errors which enter into the work.

4. The Observer. — Lastly, we come to the observer himself as the third element in making an observation. Like the external conditions, he is a variable factor; all new observers certainly are.

The observer, having put his instrument in adjustment and satisfied himself that the external conditions are favorable, should not begin work if he is seeking the highest degree of accuracy unless he considers that he himself is in his normal condition. If he is not in this condition, he introduces an

unknown disturbing element unnecessarily. He is also more liable to make mistakes in his readings and in his record. For the same reason he should not continue a series of observations too long at one time, as from fatigue the latter part of his work will not compare favorably with the first. In time-determinations, for instance, nothing is gained by observing from dark until daylight.

The observer is supposed to have no bias. A good observer, having taken all possible precautions with the adjustments of his instrument, and knowing no reason for not doing good work, will feel a certain amount of indifference towards the results obtained. The man with a theory to substantiate is rarely a good observer, unless, indeed, he regards his theory as an enemy, and not as a thing to be fondled and petted.

The greater an observer's experience, the more do his habits of observation become fixed, and the more mechanical does he become in certain parts of his work. But his judgment may be constantly at fault. Thus, with the astronomical transit he may estimate the time of a star crossing a wire in the focus of the telescope invariably too soon or invariably too late, according to the nature of his temperament. If he is doing comparison work involving micrometer bisections, he may consider the graduation mark sighted at to be exactly between the center wires of the microscope when it is constantly on the same side of the center. This fixed peculiarity, which none but experienced observers have, is known as their personal error, or *personal equation.*

In combining one observer's results with those of another observer, we must either find by special experiment the difference of their personal errors and apply it as a correction to the final result, or else eliminate it by the method of observation. Thus, in longitude work the present practice is to eliminate the effect of personal error from the final result by having the observers change places at the middle of the work.

It is always safer to eliminate the correction by the method

of observing rather than by computing for it. For though it may happen that so long as instruments and conditions are the same, the relative personal error of two observers may be constant, yet some apparently trifling change of conditions, such, for example, as illuminating the wires of the instrument differently, may cause it to be altogether changed in character.

On account of personal error, if for no other reason, it is evident that no number of sets of measures obtained in the same way by a single observer ought to be expected to furnish as good a determination of the value of a quantity as might be obtained by varying the form of making the observations or increasing the number of observers.

5. When all known corrections for instrument, for external conditions, and for peculiarities of the observer, have been applied to a direct measure, have we obtained a correct value of the quantity measured? That we cannot say. If the observation is repeated a number of times with equal care, different results will in general be obtained.

The reason why the different measures may be expected to disagree with one another has been indicated in the preceding pages. There may have been no change in the conditions of sufficient importance to attract the observer's attention when making the observations, but he may have handled his instrument differently, turned certain screws with a more or less delicate touch, and the external conditions may have been different. What the real disturbing causes were, he has no means of knowing fully. If he had, he could correct for them, and so bring the measures into accordance. Infinite knowledge alone could do this. With our limited powers, we must expect a residuum of error in our best executed measures, and, instead of certainty in our results, look only for probability.

The discrepancies from the true value due to these unexplained disturbing causes we call *errors*. These errors are *accidental*, being wholly beyond all our efforts to control. They are as likely to be in excess as in defect. If we can ferret out

the law of their operation, they cease to be classed as errors, and become corrections.

A very troublesome source of discrepancies in measured values arises from blunders made by the observer in reading his instrument or in recording his readings. Blunders from imperfect hearing, from transposition of figures and from writing one figure when another is intended, from mistaking one figure on a graduated scale for another, as 7 for 9, 3 for 8, etc., are not uncommon, — nor are mistakes of level reading of 5 divisions, of estimation of time of 1^s, and the like.

Carelessness may produce the same effect as a blunder in reading or in recording. Thus, handling a striding level roughly, or bringing a heated lamp too near it, may affect a result very seriously.

Having, then, taken all possible precautions in making the observations, and applied all possible corrections to the observed values, the resulting values, which we shall in future refer to as the *observed values*, may be assumed to contain only accidental errors. We are, then, brought face to face with the question, How shall the value of the quantity sought be found from these different observed values?

CHAPTER II

THE LAW OF ERROR

The Arithmetic Mean

6. If a quantity x is to be determined by measurement, and M is a measured value of x, then, if the observation were perfect, we should have

$$x - M = 0.$$

But since, if we make a second and a third observation, we may not find the same value as we did at first, and as we can only account for the difference on the supposition that the observations are not perfect — that is, that they are affected with certain errors — we should rather write

$$\left.\begin{array}{l} x - M_1 = \Delta_1 \\ x - M_2 = \Delta_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x - M_n = \Delta_n \end{array}\right\} \qquad (1)$$

where M_1, M_2, . . . M_n are the observed values, and Δ_1, Δ_2, . . . Δ_n are the errors of the observations.

We have here n equations and $n + 1$ unknowns. What principle shall we call to our aid to solve these equations and so find x, Δ_1, Δ_2, . . . Δ_n? In answering this question, we shall follow the order of natural development of the subject, which, in the main, is also the order of its historical development.

The value sought must be some function of the observed values, and fall between the largest and smallest of them. If the observed values are arranged according to their magnitudes, they will be found to cluster around a central value.

On first thought, the value that would be chosen as the value of x would be the central value in this arrangement if the number of observations were odd, and either of the two central values if the number were even. In other words, a plausible value of the unknown would be that observed value which had as many observed values greater than it as it had less than it. Now, since a small change in any of the observed values, other than the central value, would in general produce no change in the result, the number of observations remaining the same, this method of proceeding might be regarded as giving a plausible result, more especially if the observed values were widely discrepant.

On the other hand, the taking of the central value is objectionable, because it gives the preference to a single one of the observed values; while if these values are supposed to be equally worthy of confidence, as it is reasonable to take them in the absence of all knowledge to the contrary, each ought to exert an equal influence on the result. We may, therefore, with more reason, assume the value of x to be a *symmetrical function* x_0 of the observed values.

The simplest symmetrical function of the observed values that can be chosen as the form for x_0 is their arithmetic mean — that is,

$$x_0 = \frac{(M_1 + M_2 + M_3 \cdot \cdot \cdot M_n)}{n}$$

$$= \frac{\Sigma (M)}{n}$$

where Σ is the ordinary algebraic symbol of summation,

or $$= \frac{[M]}{n}$$

in the system of notation introduced by Gauss.

The principle may be stated as follows: *If we have n observed values of an unknown, all equally good so far as we know, the most plausible value of the unknown (best value on the whole) is the arithmetic mean of the observed values.*

It may happen that the values M_1, M_2, . . . M_n are of such nature that some other symmetrical function than the arithmetic mean will satisfy the observation equations better than will the arithmetic mean. That the arithmetic mean is *on the whole* the best form of the function may be confirmed by a comparison of results following from this hypothesis with the records of experience.

7. By adding equations (1), Art. 6, and taking the mean, we have,

$$x = \frac{[M]}{n} + \frac{[\Delta]}{n} = x_0 + \frac{[\Delta]}{n}.$$

The last term of this equation will become very small if, n being very large, the sum $[\Delta]$ of the errors remains small. Now, if, after making one observation and before making another, we re-adjust our instrument, determine anew its corrections, choose the most favorable conditions for observing, and vary the form of procedure as much as possible, it is reasonable to suppose that the disturbing influences will balance one another in the result following from the proper combination of the observed values. It may take an infinite number of trials to bring this about. In the absence of all knowledge, we cannot say that it will take less. And, reckoning blunders in reading the instrument or in recording the readings as accidental errors, an infinity of a higher order than the first may be required to eliminate them.

In other words, there being no reason to suppose that an error in excess (or positive error) is more likely to occur on the whole than an error in defect (or negative error), we may, when n is a very large number, consider $\frac{[\Delta]}{n}$ to be an infinitesimal with respect to x. We may, therefore, in this case put

$$x_0 = x;$$

that is, *when the number of observed values is very great, the arithmetic mean is the true value.*

8. From the principle of the arithmetic mean, two important inferences may be derived. For, taking the arithmetic mean, x_0, of n observed values of an unknown as the most plausible value of that unknown, we may write our *observation equations* in the form,

$$\left.\begin{array}{l} x_0 - M_1 = v_1 \\ x_0 - M_2 = v_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_0 - M_n = v_n \end{array}\right\} \qquad (1)$$

where v_1, v_2, . . . v_n are called the *residual errors* of observation, or simply the *residuals*.

(*a*) By addition,

$$n x_0 - [M] = [v],$$

and $\therefore$

$$[v] = 0\,; \qquad (2)$$

that is, *the sum of the residuals is zero;* in other words, *the sum of the positive residuals is equal to the sum of the negative residuals.*

There is a very marked correspondence between the series in which n is infinitely great and x is the *true* value, and a series in which n is finite and the arithmetic mean x_0 is taken as the best value attainable. For in the first case the sum of the *errors*, Δ, is zero, and in the second the sum of the *residuals*, v, is zero.

(*b*) Let X be any assumed value of the unknown other than the arithmetic mean, and put

$$\left.\begin{array}{l} X - M_1 = v_1' \\ X - M_2 = v_2' \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ X - M_n = v_n' \end{array}\right\} \qquad (3)$$

From equations (1) and (3), by squaring and adding,

$$[v^2] = n x_0 x_0 - 2\, x_0 [M] + [M^2],$$
$$[(v')^2] = n X X - 2\, X [M] + [M^2].$$

Hence by a simple reduction,

$$[(v')^2] = [v^2] + n\left(X - \frac{[M]}{n}\right)^2.$$

Now, $\left(X - \frac{[M]}{n}\right)^2$, being a complete square, is always positive.

$$\therefore\ [(v')^2] > [v^2];$$

that is, *the sum of the squares of the residuals* v, *found by taking the arithmetic mean, is a minimum.*

Hence the name *Method of Least Squares*, which was first given by Legendre.

The Law of Error of Observed Quantities.

9. When several independent measures of the same quantity, all equally good, have been made, it must be granted that errors in excess and errors in defect are equally likely to occur to the same amount — that is, are equally probable. Experience shows that in any well-made series of observations, small errors are likely to occur more frequently than large ones, and that there is a limit to the magnitude of the error to be expected. If, therefore, a denotes this limit or maximum error, we must consider all the errors of the series to be ranged between $+a$ and $-a$, but to be most numerous in the neighborhood of zero. Hence the probability of the occurrence of an error may be assumed to be a certain function of the error.

If the probability that an error lies between 0 and Δ be denoted by $f(\Delta)$, the probability q of an error between Δ and $\Delta + d\Delta$ is given by

$$q = f(\Delta + d\Delta) - f(\Delta) = f'(\Delta)\,d\Delta = \phi(\Delta)\,d\Delta, \text{ suppose}; \qquad (1)$$

q may be taken to be the probability of the occurrence of the error Δ, since $d\Delta$ is small.

The function $\phi(\Delta)$ is called the law of distribution of error, or simply the *law of error.*

The probability that an error falls between any assigned limits b and a is the sum of the probabilities $\phi(\Delta)\,d\Delta$ extend-

ing from b to a, and is expressed in the ordinary notation of the integral calculus by

$$\int_a^b \phi(\Delta)\, d\Delta. \tag{2}$$

Hence it follows that the probability that an error does not exceed the value a is

$$\int_{-a}^{+a} \phi(\Delta)\, d\Delta. \tag{3}$$

The form of $\phi(\Delta)$ may be determined by the aid of the principles of probability. For the probability of the occurrence of the error Δ_1 is $\phi(\Delta_1)\, d\Delta_1$, of Δ_2 is $\phi(\Delta_2)\, d\Delta_2 \ldots$ The probability, Q, of the simultaneous occurrence of the complete system of errors is the product of the respective probabilities or

$$Q = \phi(\Delta_1)\,\phi(\Delta_2) \ldots \phi(\Delta_n)\, d\Delta_1,\, d\Delta_2 \ldots d\Delta_n. \tag{4}$$

If we make Q a maximum, we shall find the most probable value of the unknown x. Now when Q is a maximum, $\log Q$ is also a maximum. Hence, differentiating with respect to x, and noting that the last terms $d\Delta_1$, $d\Delta_2 \ldots d\Delta_n$ do not depend upon x, we have,

$$\begin{aligned} 0 = \frac{d(\log Q)}{dx} &= \frac{\phi'(\Delta_1)}{\phi(\Delta_1)}\frac{d\Delta_1}{dx} + \frac{\phi'(\Delta_2)}{\phi\,\Delta_2}\frac{d\Delta_2}{dx} + \ldots + \frac{\phi'(\Delta_n)}{\phi(\Delta_n)}\frac{d\Delta_n}{dx} \\ &= \frac{\phi'(\Delta_1)}{\Delta_1\,\phi(\Delta_1)}\Delta_1 + \frac{\phi'(\Delta_2)}{\Delta_2\phi(\Delta_2)}\Delta_2 + \ldots + \frac{\phi'(\Delta_n)}{\Delta_n\,\phi(\Delta_n)}\Delta_n, \end{aligned} \tag{5}$$

since from Eq. 1, Art. 6,

$$\frac{d\Delta_1}{dx} = \frac{d\Delta_2}{dx} = \ldots = \frac{d\Delta_n}{dx} = 1.$$

But from the principle of the arithmetic mean, when the number of observed values is very great,

$$\Delta_1 + \Delta_2 + \ldots + \Delta_n = 0. \tag{6}$$

Also, since equations (5) and (6) must be simultaneously satisfied by the same value of the unknown, we necessarily have,

$$\frac{\phi'(\Delta_1)}{\Delta_1\,\phi(\Delta_1)} = \frac{\phi'(\Delta_2)}{\Delta_2\,\phi(\Delta_2)} = \ldots = \frac{\phi'(\Delta_n)}{\Delta_n\,\phi(\Delta_n)} = k, \text{ suppose.}$$

Hence for any arbitrary value Δ,

$$\frac{\phi'(\Delta)}{\Delta\,\phi(\Delta)} = k.$$

Clearing of fractions and integrating,

$$\phi(\Delta) = ce^{\frac{1}{2}k\Delta^2},$$

where e is the base of the system of natural logarithms and c is a constant.

Again, since Q is to be a maximum, $\frac{d^2Q}{dx^2}$ or $\frac{d^2(\log Q)}{dx^2}$ must be negative. Now,

$$Q = c^n e^{\frac{k}{2}(\Delta_1{}^2+\Delta_2{}^2+\ldots)}\,d\Delta_1\,d\Delta_2 \ldots$$

$$\frac{d(\log Q)}{dx} = k(\Delta_1 + \Delta_2 + \ldots)$$

$$\frac{d^2(\log Q)}{dx^2} = kn.$$

Hence, since n is positive, k must be negative, and putting $\frac{1}{2}k = -h^2$, we have

$$\phi(\Delta) = ce^{-h^2\Delta^2},$$

the law of error sought.

10. In this expression there are two symbols undetermined, c and h. To find c. Since it is certain that all of the errors lie between the maximum errors $+a$ and $-a$, we have

$$c\int_{-a}^{+a} e^{-h^2\Delta^2}d\Delta = 1.$$

But as the values of a are different for different kinds of observations, and as we cannot in general assign these values definitely, we must take $+\infty$ and $-\infty$ as the extreme limits of error, so that c is found from

$$c\int_{-\infty}^{+\infty} e^{-h^2\Delta^2}d\Delta = 1,$$

and hence
$$c = \frac{h}{\sqrt{\pi}},$$

and the law of error may be written,

$$\phi(\Delta) = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$$

As regards h, it is evident that for $e^{-h^2\Delta^2}$ to be a possible quantity, h must be an abstract number. Hence, $\frac{1}{h}$ is a quantity expressed in the same unit of measure as x.

Also from the form of the function $\phi(\Delta)$ it is evident that the probability of an error Δ will be the larger, the smaller h is, and *vice versa*. Hence, h is a test of the quality of observations of different series. It was named by Gauss *the measure of precision.*

In practice, it is more convenient to compare the precision of different series of observations by other methods.

11. Proof of the Law of Error on Hagen's (Young's) Hypothesis. — Various proofs of the law of error have been derived. Each is open to some theoretical objection. The following proof of Hagen's hypothesis starts with a clear and definite statement of the assumed nature of an accidental error, namely, that it is the algebraic sum of an infinite number of independent infinitesimal element errors, each of which is as likely to be positive as negative. The law of error being derived directly from this hypothesis, it is clear that if an observer wishes to put the errors of observation into this accidental class, and, therefore, to make them easy to eliminate, he must use such instruments and methods as will make the errors conform as closely as possible to this definition of an accidental error. It is because Hagen's proof thus indicates clearly to the observer the standard toward which he should struggle, that it is here given in addition to the complete and independent proof in the preceding article.

An accidental error of observation does not result from a

single cause. Thus, in reading an angle with a theodolite, the error in the value found is the result of imperfect adjustment of the instrument, of various atmospheric changes, of want of precision in the observer's method of handling the instrument, etc. Each of these influences may be taken as the result of numerous other influences. Thus, the first mentioned may include errors of collimation, of level, etc. Each of these in turn may be taken as resulting from other influences, and so on. The final influences, or element errors, as they may be called, must be assumed to be independent of one another, and each as likely to make the resultant error too large as too small — that is, as likely to be positive as negative. The number of these element errors being very great, we may, from the impossibility of assigning the limit, consider it as infinite in any case. Each element error must consequently be an infinitesimal, and for greater simplicity we may take those occurring in any one series as of the same numerical magnitude. Hence we conclude that an error of observation may be assumed to be the algebraic sum of a very great number of independent infinitesimal element errors ϵ, all equal in magnitude, but as likely to be positive as negative.

Let the number of these element errors be denoted by $2n$, as the generality of the demonstration will not be affected by supposing this infinitely great number to be even. If all of the element errors are $+$, the error $2n\epsilon$ results, and this can occur in but one way; if all but one are $+$, the error $(2n-2)\epsilon$ results, and this can occur in $2n$ ways; and generally, if $n+m$ are $+$, and $n-m$ are $-$, the error $2m\epsilon$ results, and this can occur in $\dfrac{2n(2n-1)\cdots(n+m+1)}{1\,2\cdots(n-m)}$ ways.* Hence the numbers expressing the relative frequency of the errors (that is, the number of times they may be expected to occur) are equal to the coefficients in the development of the $2n$th power of any binomial.

* See Todhunter's or Newcomb's *Algebra*.

The element errors, infinite in number, being infinitely small in comparison with the actual errors of observation, these latter may consequently be assumed to be continuous from 0 to $2n\epsilon$, the maximum error. If, therefore, Δ denotes the error in which $n + m$ $+\epsilon$'s and $n - m$ $-\epsilon$'s occur, and $\Delta + d\Delta$ denotes the consecutive error in which $n + m + 1$ $+\epsilon$'s and $n - m - 1$ $-\epsilon$'s occur, we have

$$\Delta = 2m\epsilon$$

$$\Delta + d\Delta = (2m + 2)\epsilon,$$

and therefore,

$$\Delta = md\Delta.$$

Calling f the relative frequency of the error Δ, and $f + df$ that of the consecutive error $\Delta + d\Delta$, we have

$$f = \frac{2n(2n-1)\cdots(n+m+1)}{n-m!},$$

$$f + df = f\frac{2n(2n-1)\cdots(n+m+2)}{n-m-1!}.$$

Hence, by division,

$$\frac{f + df}{f} = \frac{n-m}{n+m+1},$$

or

$$\frac{df}{f} = -\frac{2m+1}{n+m+1}$$

$$= -\frac{2\Delta + d\Delta}{nd\Delta + \Delta + d\Delta}.$$

Now, since $d\Delta$ is infinitely small in comparison with Δ, we may write

$$\frac{df}{f} = -\frac{2\Delta}{nd\Delta + \Delta}.$$

Also, since df is infinitely small in comparison with f, 2Δ is with respect to $nd\Delta + \Delta$, and we may neglect Δ in the denominator in comparison with $nd\Delta$. We have, therefore,

$$\frac{df}{f} = -\frac{2\Delta}{nd\Delta}.$$

And since Δ is infinitely small in comparison with $nd\Delta$, and $d\Delta$ is infinitely small in comparison with Δ, it follows that n must be an infinity of the second order. It is, therefore, of a magnitude comparable with $\frac{1}{(d\Delta)^2}$, and hence, $n\,(d\Delta)^2$ must be a finite constant. Calling this constant $\frac{1}{h^2}$, we have

$$\frac{df}{f} = -\,2\,h^2\Delta\,d\Delta.$$

Integrating and denoting the value of f, when $\Delta = 0$, by f_0,

$$f = f_0 e^{-h^2\Delta^2}.$$

The errors being separated by the intervals $d\Delta$, so that 0, $d\Delta$, . . . Δ, $\Delta + d\Delta$ . . . are the errors in order of magnitude, we must, in order to make the system consistent with the definition of probability, and therefore continuous, consider not so much the relative frequency of the detached errors as the relative frequency of the errors within certain limits.

Now, by the definition of probability, the probability of an error between the limits Δ and $\Delta + d\Delta$ is represented by a fraction whose numerator is the number of errors which fall between Δ and $\Delta + d\Delta$, and denominator the total number of errors committed. If we denote this probability by $\phi\,(\Delta)$ we may write

$$\begin{aligned}\phi\,(\Delta) &= \frac{f}{\Sigma f}\\ &= ce^{-h^2\Delta^2},\end{aligned}$$

where c is a constant, Σf being necessarily a constant for the same series of observations.

12. The Principle of Least Squares. — Let us return to 9, Eq. 4.

If the observed values are of the same quality throughout, h is constant and the product becomes $c^n e^{-h^2[\Delta^2]}$. This product is evidently a maximum when $[\Delta^2]$ is a minimum; that is, *if we assume that each of a very large number of observed values of a*

quantity is of the same quality, the most probable value of the quantity is found by making the sum of the squares of the errors a minimum.

If the observed values are not of the same quality, h is different for the different observations, and the most probable value of the unknown would be found from the maximum value of $e^{-[h^2\Delta^2]}$; that is, from the minimum value of $[h^2\Delta^2]$. Thus, *if each of a large number of observed values of a quantity is of different quality, the most probable value of the quantity is found by multiplying each error of observation by its h, and making the sum of the squares of the products a minimum.*

The Law of Error of a Linear Function of Independently Observed Quantities.

13. We have found the law of error in the case of a quantity directly observed, and which may be a function of one or more unknowns. There remains the question as to the form the law of error assumes in the case of a quantity, F, which is a linear function of several independently observed quantities, M_1, M_2, . . . M_n; that is, when

$$F = a_1M_1 + a_2M_2 + \cdot \cdot \cdot + a_nM_n,$$

where a_1, a_2, . . . are all constants.

For simplicity in writing, consider two observed quantities, M_1, M_2, only, and let h_1, h_2 be their measures of precision. The probability of the simultaneous occurrence of the errors Δ_1 in M_1 and Δ_2 in M_2 is

$$\frac{h_1h_2}{\pi} e^{-h_1^2\Delta_1^2 - h_2^2\Delta_2^2} d\Delta_1 d\Delta_2. \tag{1}$$

Now, an error Δ_1 in M_1 and an error Δ_2 in M_2 produce an error in F, according to the relation

$$\Delta = a_1\Delta_1 + a_2\Delta_2, \tag{2}$$

and this relation can always be satisfied by combining any value

of Δ_2 with all values of Δ_1 ranging from $-\infty$ to $+\infty$. The probability, therefore, of an error Δ in F may be written,

$$\phi(\Delta)d\Delta = \frac{h_1 h_2}{\pi} d\Delta_2 \int_{-\infty}^{+\infty} e^{-h_1^2\Delta_1^2 - h_2^2\Delta_2^2} d\Delta_1.$$

But from (2), and since Δ_2 is independent of Δ_1,

$$d\Delta = a_2 d\Delta_2.$$

Hence,

$$\phi(\Delta)\,d\Delta = \frac{h_1 h_2}{\pi}\,\frac{d\Delta}{a_2}\int_{-\infty}^{+\infty} e^{-h_1^2\Delta_1^2 - h_2^2\left(\frac{\Delta - a_1\Delta_1}{a^2}\right)^2} d\Delta_1$$

$$= \frac{h_1 h_2}{\pi}\frac{d\Delta}{a^2} e^{-\frac{h_1^2 h_2^2}{h_1^2 a_2^2 + h_2^2 a_1^2}\Delta^2} \int_{-\infty}^{+\infty} e^{-\frac{h_1^2 a_2^2 + h_2^2 a_1^2}{a_2^2}\left(\Delta_1 - \frac{h_2^2 a_1}{h_1^2 a_2^2 + h_2^2 a_1^2}\Delta\right)^2} d\Delta_1$$

$$= \frac{h_1 h_2}{\sqrt{(h_1^2 a_2^2 + h_2^2 a_1^2)\,\pi}}\, e^{-\frac{h_1^2 h_2^2}{h_1^2 a_2^2 + h_2 a_1^2}\Delta^2} d\Delta,$$

which is of the form

$$\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta.$$

That is, *the law of error of the function F is the same as that of* the independently measured quantities M_1, M_2.

The precision of the function F is found from

$$h^2 = \frac{h_1^2 h_2^2}{h_1^2 a_2^2 + h_2^2 a_1^2};$$

that is, from

$$\frac{1}{h^2} = \frac{a_1^2}{h_1^2} + \frac{a_2^2}{h_2^2} = \left[\frac{a^2}{h^2}\right].$$

This theorem is one of the most important in the method of least squares, and will be often referred to.

Ex.—To find the precision of the arithmetic mean of n equally well-observed values of a quantity:

We have, $\quad F = \frac{1}{n}(M_1 + M_2 + \cdots + M_n).$

Let $\quad h_0 =$ precision of the arithmetic mean;
$\quad h =$ precision of each observed value.

Then $\quad \frac{1}{h_0^2} = \frac{1}{n^2}\left(\frac{1}{h^2} + \frac{1}{h^2} + \cdots \text{ to } n \text{ terms}\right),$

or $$h_0 = \sqrt{n}h.$$

That is, *the precision of the arithmetic mean of n observations is $\sqrt{n}$ times that of a single observation.*

14. Comparison of the Accuracy of Different Series of Observations.—We have seen that the measure of precision h affords a test of the relative accuracy of different series of observations. This test was suggested by the form of the law of error, and is naturally the first that would be chosen for that purpose.

The Mean-Square Error. — On account of the inconvenience of computing h, Gauss suggested the *mean-square* or "to be feared" error as a test of quality. This is defined as a quantity μ whose square is equal to the mean or average of the squares of the individual errors, or when n is very large.

$$\mu^2 = \frac{(\Delta^2 + \Delta_2^{\ 2} + \cdots + \Delta_n^{\ 2})}{n} = \frac{[\Delta^2]}{n}.$$

To find the relation between h and μ. The probability of the occurrence of an error Δ, that is, of an error between Δ and $\Delta + d\Delta$, is $\phi(\Delta)\,d\Delta$. The number of errors in the series being n, the sum of the squares of the errors in the same interval will be $n\Delta^2\phi(\Delta)\,d\Delta$, and the sum of the squares of the errors between the limits of error $+a$ and $-a$ will be for a continuous series.

$$n\int_{-a}^{+a}\Delta^2\phi(\Delta)\,d\Delta.$$

Extending the limits of error $\pm a$ to $\pm\infty$, n being very large, we have

$$n\mu^2 = \frac{nh}{\sqrt{\pi}}\int_{-\infty}^{\infty}\Delta^2 e^{-h^2\Delta^2}d\Delta = \frac{n}{2h^2},$$

or $$\mu = \frac{1}{h\sqrt{2}}.$$

15. The Probable Error.—The most common method in use in this country of determining the relative precision of different

series of observations is by comparing errors which occupy the same relative position in the different series when the errors are arranged in order of magnitude. The errors which occupy the middle places in each series are, for greater convenience, the ones chosen.

Let the errors in a series, arranged in order of magnitude, be

$$\pm 2a, \cdots \pm r, \cdots 0,$$

each error being written as many times as it occurs; then we give to that error r which occupies the middle place, and which has as many errors numerically greater than it as it has errors less than it, the name of *probable error.* If, therefore, n is the total number of errors, the number lying between $+r$ and $-r$ is $n/2$, and the number outside these limits is also $n/2$. In other words, the probability that the error of a single observation in any system will fall between the limits $+r$ and $-r$ is $1/2$, and the probability that it will fall outside these limits is also $1/2$. We have, therefore,

$$\frac{h}{\sqrt{\pi}}\int_{-r}^{+r} e^{-h^2\Delta^2}d\Delta = \frac{1}{2},$$

from which to find r.

If we put $h\Delta = t$, and the value $t = \rho$ corresponds to $\Delta = r$, then

$$\frac{2}{\sqrt{\pi}}\int_0^{\rho} e^{-t^2}dt = \frac{1}{2}.$$

Expanding the integral in a series (see Art. 22), we shall find that approximately the resulting equation is satisfied by

$$\rho = 0.47694.$$

Now, since

$$hr = \rho = 0.47694 \text{ and } h\mu\sqrt{2} = 1,$$

it follows that

$$\begin{aligned} r &= 0.6745\,\mu \\ &= \frac{2}{3}\mu \text{ roughly.} \end{aligned}$$

Hence, to find the probable error, we compute first the mean-square error and multiply it by 0.6745.

As a check, the error which occupies the middle place in the series of errors arranged in order of magnitude may be found. It will be nearly equal to the computed value, if the seriesis of considerable length.

It is to be clearly understood that the term probable error does not mean that that error is more probable than any other, but only that in a future observation the probability of committing an error greater than the probable error is equal to the probability of committing an error less than the probable error. Indeed, of any single error the most probable is zero. Thus the probability of the error zero is to that of the probable error r as

$$\frac{h}{\sqrt{\pi}} : \frac{h}{\sqrt{\pi}} e^{-h^2r^2},$$

or

$$1 : e^{-(0.47694)^2},$$

or

$$1 : 0.8.$$

The idea of probable error is due to Bessel (*Berlin. Astron. Jahrb.*, 1818). The name is not a good one, on account of the word "probable" being used in a sense altogether different from its ordinary signification. It would be better to use the term *critical error*, for example, as suggested by De Morgan, or *median error*, as proposed by Cournot.

16. The Average Error.—It naturally occurs as a third test of the precision of different series of observations, to take the mean of all the positive errors and the mean of all the negative errors, and then, since in a large number of observations there will be nearly the same number of each kind, to take the mean of the two results without regard to sign. This gives what may be termed the *average error*. It is usually denoted by the Greek letter η, so that

$$\eta = \frac{[\Delta}{n},$$

where $[\Delta$ is the arithmetic sum of the errors.

An expression for η in terms of the mean-square error μ may be found as follows. The number of errors between Δ and $\Delta + d\Delta$ is

$$n\phi(\Delta)\,d\Delta,$$

and the sum of the positive errors in the series is

$$n\int_0^{\infty}\Delta\phi(\Delta)\,d\Delta.$$

The sum of the negative errors being the same, the sum of all the errors is

$$2n\int_0^{+\infty}\Delta\phi(\Delta)\,d\Delta.$$

Hence
$$\eta = 2\int_0^{+\infty}\Delta\phi(\Delta)\,d\Delta$$
$$= \frac{2h}{\sqrt{\pi}}\int_0^{+\infty}\Delta e^{-h^2\Delta^2}d\Delta$$
$$= \frac{1}{h\sqrt{\pi}} = \mu\sqrt{\frac{2}{\pi}},$$

the relation required.

The average error may, as stated above, be directly used as a test of the relative accuracy of different series of observations. The general custom is, however, to employ it as a stepping-stone to find the mean-square and probable errors. This can be done, for the reason that it is more easy to compute $[\Delta$ than $[\Delta^2]$.

17. The formulas for μ and r computed in this way are as follows. From the last equation preceding

$$\mu = \sqrt{\frac{\pi}{2}}\,\eta$$
$$= 1.2533\,\frac{[\Delta}{n},$$

and from Art. 14,
$$r = 0.6745\,\mu$$
$$= 0.8453\,\frac{[\Delta}{n}.$$

The relations connecting μ, r, and η are easily remembered in the following form :

$$\mu\sqrt{2} = \frac{r}{\rho} = \sqrt{\pi}\eta.$$

These relations may also be conveniently arranged in tabular form :

	μ	r	η
$\mu =$	1.0000	1.4826	1.2533
$r =$	0.6745	1.0000	0.8453
$\eta =$	0.7979	1.1829	1.0000

The p. e. which is to be used as a measure of accuracy may be computed from the sum of the squares of the errors and also from the sum of the errors without regard to sign. The question then arises, which of the two methods will give the better result? It may be stated without here setting forth the proof, that the value of the p. e. computed from the squares is somewhat more trustworthy than it is when derived from the first powers of the errors.

18. Whether we should use the m. s. e. or the p. e. in stating the precision is largely a matter of taste. Gauss says : "The so-called probable error, since it depends on hypothesis, I, for my part, would like to see altogether banished ; it may, however, be computed from the mean by multiplying by 0.6744897." On the other hand, the International Committee of Weights and Measures decided in favor of the probable error : "It has been thought best in this work that the measure of precision of the values obtained should always be referred to the probable error computed from Gauss' formula, and not to the mean error." (*Procès Verbaux*, 1879, p. 77.)

In the United States, in the Naval Observatory, the Coast Survey, the Engineer Corps, and the principal observatories, the p. e. is used altogether. So, too, in Great Britain, in the Greenwich Observatory, the Ordnance Survey, etc. In the G. T.

Survey of India the m. s. e. is used, for the reason given by Gauss above. Among German geodeticians and astronomers the m. s. e. is very generally employed.

The p. e. has a definite meaning, namely, that the chances are even for and against a given error being greater or less than the corresponding p. e. This frequently furnishes a convenient test as to whether the errors of a given series of observations are distributed according to the assumed law of error.

The p. e. will be used in the text of this book nearly always.

The Probability Curve.

19. The principles laid down in the preceding articles may be illustrated geometrically as follows:

We have seen that in a series of observations the probability of an error Δ, that is, that an error will lie between the values Δ and $\Delta + d\Delta$, is given by the expression (Art. 13)

$$\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2} d\Delta.$$

Now, if O is the origin of coördinates, and a series of errors, Δ, are represented by the distances from O along the axis of abscissas OX, positive errors being taken to the right of O and negative errors to the left, then the probability, in a future observation, of an error falling between Δ and $\Delta + d\Delta$, will be represented by the rectangle whose height is $\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$ and width $d\Delta$, or, more strictly, by the ratio of this rectangle to the sum of all such rectangles between the extreme limits of error. This sum we have for convenience already denoted by unity.

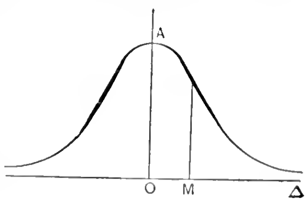

Fig. 1.

Hence, for a series of observations whose quality is known, by giving to Δ all values from $+\infty$ to $-\infty$ and drawing the

corresponding ordinates, we shall have a continuous curve whose equation may be written

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}.$$

This curve is called the probability curve.

20. To Trace the Form of the Curve. — Since Δ enters to the second power, and y to the first power, the curve is symmetrical with respect to the axis of y, and the form of the equation shows that it lies altogether on one side of the axis of Δ. Also, when $\Delta = 0$, $\frac{dy}{d\Delta} = 0$; that is, the tangent at the vertex is parallel to the axis of x.

As Δ increases from 0 the values of y continually decrease. When $\Delta = \pm \infty$, then

$$y = 0 \quad \text{and} \quad dy/d\Delta = 0,$$

showing that the axis of Δ is an asymptote.

Again, since

$$\frac{d^2y}{d\Delta^2} = \frac{2h^3}{\sqrt{\pi}} e^{-h^2\Delta^2} (2h^2\Delta^2 - 1),$$

there is a point of inflection when

$$\Delta = \frac{1}{h\sqrt{2}} = \mu,$$

and the m. s. e. is therefore the abscissa of the point of inflection. Also, when $\Delta = 0$, $d^2y/d\Delta^2$ is negative, showing that the ordinate at the vertex is the maximum ordinate. Hence the curve is of the form indicated in Fig. 1, OA representing the maximum ordinate and OM the m. s. e.

The values of h, that is, of $1/\mu\sqrt{2}$, being different for different series of observations, the form of the curve will change for each series, and the curve may be plotted to scale from values of y corresponding to assumed values of Δ.

In plotting the curve, since the maximum ordinate at the vertex $h\sqrt{\pi}$ enters as a factor into the values of each of the other ordinates, its value may be arbitrarily assumed. We may there-

fore adopt a scale for plotting the ordinates different from the scale by which the abscissas are plotted, in order to show the curve more clearly.

The form of the curve is in accordance with the principles already laid down in deducing the law of error, and could have been derived from them directly. Thus, that small errors are more probable than large, is indicated by the element rectangle areas being greater for values of Δ near zero than for values more distant; that very large errors have a very small probability is indicated by the asymptotic form of the curve; and that positive and negative errors are equally probable, is indicated by its symmetrical form with respect to the axis of y.

21. The area of the curve of probability is the sum of the rectangles $\frac{h}{\sqrt{\pi}}\, e^{-h^2\Delta^2}d\Delta$, for values of Δ extending from $+\infty$ to $-\infty$, and may be denoted by unity. If, then, we represent by their area the total number of errors that occur in a series of observations, it follows from the definition of probability that the area included between certain assigned limits will represent the number of errors to be expected in the series between the values of those limits.

Thus, if O is the origin, the area to the right of OA would represent the number of positive errors, and the area to the left of OA the number of negative errors. The area $OPP'A$ would represent the number of positive errors less than OP, the area $PRR'P'$ the number that lie between OP and OR.

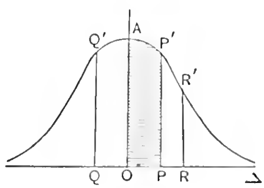

Fig. 2.

If the area $AOPP'$ is equal to one-half the total area AOX, then the number of positive errors less than OP would be equal to the number greater than OP. Hence OP would represent the probable error. If OQ be taken equal to OP, the area $PQQ'P'$ would represent the number of errors numerically less than the probable error.

The average error η is evidently represented by the abscissa of the center of gravity of either the positive or negative half of the probability curve.

The Law of Error Applied to an Actual Series of Observations.

We here bridge over the gulf between the ideal series from which we have derived the law of error, and the actual series with which we have to deal in practical work, and which can only be expected to come partially within the range of the law constructed for the ideal.

22. Effect of Extending the Limits of Error to $\pm\,\infty$. — The expression $\dfrac{h}{\sqrt{\pi}}\,e^{-h^2\Delta^2}d\Delta$ gives the value of the probability of an error between Δ and $\Delta + d\Delta$ in an ideal series of observations where the values are continuous between limits infinitely great. In all actual series the possible error is included within certain finite limits, and the probability of the occurrence of an error beyond those limits is zero. Practically, however, the extension of the limits of error to $\pm\,\infty$ can make no appreciable difference in either case, as the function $\phi\,(\Delta)$ decreases so rapidly that we can regard it as infinitesimal for large values of Δ; in other words, the greater number of errors is in the neighborhood of zero, and therefore the most important part is the part covered by both. This has been illustrated geometrically in the discussion of the probability curve, and will now be developed from another point of view.

The probability of the occurrence of an error not greater than a in a series of observations is, since the error must lie between $+\,a$ and $-\,a$,

$$\frac{h}{\sqrt{\pi}}\int_{-a}^{+a} e^{-h^2\Delta^2}d\Delta \text{ (see Art. 13).}$$

This is the same as the area of the curve of probability between the limits $+a$ and $-a$, the total area being unity (Art. 13). Change the variable by placing $h\Delta = t$. The expression then becomes,

$$\frac{\Delta}{\sqrt{\pi}}\int_{-ha}^{ha} e^{-t^2}dt \quad \text{or} \quad \frac{2}{\sqrt{\pi}}\int_{0}^{ha} e^{-t^2}dt,$$

and is usually denoted by the symbol $\Theta\,(t)$.

If, as in Art. 14, the value $t = \rho$ corresponds to $\Delta = r$, we have finally, by eliminating h,

$$\Theta\,(t) = \frac{2}{\sqrt{\pi}}\int_0^{\rho\frac{a}{r}} e^{-t^2}dt.$$

The value of this integral cannot be expressed exactly in a finite form, but may be found approximately as follows:

Expanding e^{-t^2} in a series, and integrating each term separately, we have,

$$\int_0^t e^{-t^2}dt = \int_0^t\left(1 - \frac{t^2}{1} + \frac{t^4}{1.2} - \ldots\right)dt$$
$$= t - \frac{t^3}{3} + \frac{1}{1.2}\,\frac{t^5}{5} - \ldots$$

This series is convergent for all values of t, but the convergence is only rapid enough for small values of t.

For large values of t it is better to proceed as follows:

Integrating by parts,

$$\int e^{-t^2}dt = \int -\frac{1}{2\,t}\,de^{-t^2}$$
$$= -\frac{1}{2\,t}e^{-t^2} - \frac{1}{2}\int\frac{e^{-t^2}}{t^2}\,dt$$
$$= -\frac{1}{2\,t}e^{-t^2} + \frac{1}{2^2t^3}e^{-t^2} + \frac{1\cdot 3}{2^2}\int\frac{e^{-t^2}}{t^4}\,dt.$$

Hence $\displaystyle\int_a^t e^{-t^2}dt = \frac{e^{-t^2}}{2\,a}\left\{1 - \frac{1}{2\,t^2} + \frac{1\cdot 3}{(2\,t^2)^2} - \frac{1\cdot 3\cdot 5}{(2\,t^2)^3} + \ldots\right\}$

But $\displaystyle\int_0^t e^{-t^2}dt = \int_0^t e^{-t^2}dt - \int_t^\infty e^{-t^2}dt$

$$= \frac{\sqrt{\pi}}{2} - \int_t^\infty e^{-t^2}dt.$$

$\therefore$ finally,

$$\int_0^t e^{-t^2}dt = \frac{\sqrt{\pi}}{2} - \frac{e^{-t^2}}{2\,t}\left\{1 - \frac{1}{2\,t^2} + \frac{1\cdot 3}{(2\,t^2)^2} - \frac{1\cdot 3\cdot 5}{(2\,t^2)^3} + \ldots\right\}$$

23. Approximate values of the expression $\int_0^t e^{-t^2}dt$ may be computed from the above formulas for any numerical value of t.

In Table I (Art. 213) will be found the values of the function $\Theta(t)$ corresponding to the argument a/r. The reason for arranging the table in this way is that it is more convenient to compute $\frac{a}{r}$ than $\rho\frac{a}{r}$, where

$$\rho = 0.47696.$$

The probability that an error exceeds a certain error a is $1 - \Theta(t)$, and may be found from Table I by deducting the tabular value from unity. Thus we have the probability that a is greater than r is 0.5, than $2r$ is 0.177, than $3r$ is 0.043, than $4r$ is 0.007, than $5r$ is 0.001, than $6r$ is 0.0001.

Hence, in 10,000 observations we should expect only one error greater than $6r$, in 1000 only one greater than $5r$, in 100 only one greater than $4r$, and in 25 only one greater than $3r$. If in any set of observations we found results much at variance with these, we could assume that they arose from some unusual cause, and should, therefore, be specially examined. As in practice the number of observations in any case is usually under 100, we are eminently safe in taking the maximum error at about $5r$ or 3μ.

Experience indicates that in general the curve representing the true law of error for a given series of observations departs but little from the Gaussian curve. The actual curve for a given series may quickly be compared with the Gaussian curve by the use of Table I. The degree of departure from the Gaussian curve necessarily indicates the extent to which the facts differ from the assumption of an infinite number of sources of infinitesimal errors.

If a set of observations shows a marked divergence from this law, a rigid examination will reveal the necessity, in general, of applying some hitherto unknown correction. Thus, in the earlier differential comparisons of the compensating base-apparatus

of the United States Lake Survey with the standard bar packed in ice, the observed differences did not follow the law of error, as it was fair to suppose that they should, the bars being compensating. There was instead a regular daily cycle; some one source of error so far exceeded the others that it overshadowed them. A study of the results was made, and the law of daily change discovered, which gave a means of applying a further correction. The work done later, after taking account of this new correction, showed nothing unusual.

24. General Conclusion. — On the whole, though we cannot say that the formula $\frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}$ will truly represent the law of error in any given series of observations, we can say that it is a close approximation.

When in a series of observations we have exhausted all of our resources in finding the corrections, and have applied them to the measured values, the residuum of error may fairly be supposed to have arisen from many sources; and we conclude from the foregoing investigations that, of any one single law, the best to which we can consider the residual errors subject, and the best to be applied to a set of observations not yet made, is the exponential law of error.

The general theorem of Art. 12 may therefore be applied to a limited series and be written: *If the observed values of a quantity are of different quality, the most probable value is found by dividing each residual error by the probable error and making the sum of the squares of the quotients a minimum; if of the same quality, the most probable value is the arithmetic mean of the observed values.*

Classification of Observations

25. For purposes of reduction, observations may be divided into two classes — those which are independent, being subject to no conditions except those fixed by the observations themselves; and those which are subject to certain conditions outside of the

observations, as well as to the conditions fixed by the observations. In the former class, before the observations are made, any one assumed set of values is as likely as any other; in the latter no set of values can be assumed to satisfy approximately the observation equations which does not exactly satisfy the *à priori* conditions.

For example, suppose that at a station O the angles AOB, AOC, are measured. If the measures of each angle are independent of those of the other, the angles are found directly.

The angle BOC could be determined from the relation

$$AOC = AOB + BOC.$$

The unknown in this case may be said to be observed indirectly, and therefore independent observations may be classed as *direct* and *indirect*. The former class is a special class of the latter.

But if the angle BOC is observed directly as well as AOB, AOC, then these angles are no longer independent, but are subject to the condition that when adjusted

$$AOC = AOB + BOC,$$

and no set of values can be assumed as possible which does not exactly satisfy this condition.

The observations in this case are said to be *conditioned.* Though we have, therefore, strictly speaking, only two classes of observations, we shall, for simplicity, divide the first into two, and consider in order the adjustment of

(1) Direct observations of one unknown.

(2) Indirect observations of several independent unknowns.

(3) Conditioned observations.

CHAPTER III

ADJUSTMENT OF DIRECT OBSERVATIONS OF ONE UNKNOWN QUANTITY

In the application of the ideal formulas of Chapter II to an actual series of observations, we shall begin with a single quantity which has been directly observed. We shall consider two cases — first, when all of the observed values are of equal quality, and, next, when they are not all of equal quality.

There are in all cases two quantities to be found — first the most probable value of the unknown itself, and next the precision of this value.

A. Observed Values of Equal Quality.

26. The Most Probable Value; the Arithmetic Mean.—We have seen that in a series of directly observed values M_1, M_2, . . . M_n, of equal quality, the most probable value x_0 of the observed quantity is found by taking the arithmetic mean of these values; that is,

$$x_0 = [M]/n. \qquad (1)$$

It has also been shown that the same result will follow by making the sum of the squares of the residual errors a minimum. (Art. 12.)

As the observed values M are often numerically large and not widely different, the arithmetical work of finding the mean may be shortened as follows:

A cursory examination of the observations will show about what the mean value x_0 must be. Let X' denote this approximate value of x_0, which may conveniently be taken some round number. Subtract each of the observed values M_1, M_2, . . . M_n

in succession from X' and call the differences $l_1, l_2, \ldots l_n$ respectively. Then,

$$\begin{aligned} X' - M_1 &= l_1, \\ X' - M_2 &= l_2, \\ X' - M_n &= l_n. \end{aligned} \tag{2}$$

By addition, $$nX' - [M] = [l].$$

But $$x_0 = \frac{[M]}{n}.$$

$$\begin{aligned} \therefore\ x_0 &= X' + \frac{[l]}{n} \\ &= X' + x', \text{ suppose.} \end{aligned} \tag{3}$$

Hence all that we have to do is to take the mean x' of the small quantities, $l_1, l_2, \ldots l_n$, and add the assumed value X' to the result.

27. Control of the Arithmetic Mean. — In least-square computations, it is important to have a check or control of the numerical work. This is specially desirable when a computation takes several weeks, or it may be months, to complete it. In long computations it is better for two computers to work together, using different methods whenever possible, and to compare results at intervals. But even this is not an absolute safeguard against mistakes, as it sometimes happens that both make the same slip, as, for example, writing + for −, or *vice versa*. Hence, even if the computation is made in duplicate, it is advisable to carry through an independent check which may be referred to occasionally. In computations not duplicated a control is essential.

A control of the accuracy of the arithmetic mean of a set of observed values of the same quantity is afforded by the relation

$$[v] = 0;$$

that is, that the sum of the positive residuals should be equal to the sum of the negative residuals.

If, however, in finding the arithmetic mean, the sum $[M]$ of the observed quantities was not exactly divisible by their number n, the sums of the positive and negative residuals would not

be equal, but the amount of the discrepancy could easily be estimated and allowed for. For if the value of the mean taken were too large by a certain amount, the positive residuals would each be too large, and the negative residuals too small, by that amount. Hence the discrepancy to be expected would be n times the amount that the approximate quotient taken as the mean differed from the exact quotient.

Ex. — In the telegraphic determination of the difference of longitude between St. Paul and Duluth, Minn., June 15, 1871, the following were the corrections found for chronometer Bond No. 176 at 15 h. 51 m. sidereal time from the observations of 21 time stars. (*Report Chief of Engineers, U. S. A.*, 1871.)

M	v		v^2
s	s	s	
− 8.78	+ 0.04		0.0016
.76	+ .02		4
.85	+ .11		121
.78	+ .04		16
.51		− 0.23	529
.64		− .10	100
.68		− .06	36
.63		− .11	121
.58		− .16	256
.80	+ .06		36
.75	+ .01		1
.78	+ .04		16
.96	+ .22		484
.64		− .10	100
.65		− .09	81
.83	+ .09		81
.70		− .04	16
.64		− 0.10	100
.79	+ .05		25
.90	+ .16		256
− 8.93	+ 0.19		0.0361
Mean, − 8.74	+ 1.03 $[v] = 2.02$	− 0.99	$[v^2] = .02756$

Taking the observations as of equal precision, we find the arithmetic mean to be − 8.74. This is the most probable value of the correction.

The residuals v are found by subtracting each observed value from the most probable value according to the relation,

$$X' - M = v.$$

They are written in two columns for convenience in applying the check,

$$[v] = 0.$$

The true mean may be derived by subtracting the mean of the residuals from the approximate mean.

28. It was desired in this case to secure a mean which is correct in the second decimal place, or, in other words, is not in error by more than 5 in the third decimal place. This has obviously been secured when the difference between the sums of the + and − residuals is less than (0.005) (21) = 0.105.

In general, *a mean is correct to a given decimal place when the difference between the sums of the + and − residuals is less, expressed in units of that decimal place, than one-half of the number of quantities of which the mean is sought.*

If the difference of the sums of the + and − residuals be found to be too great, an error has been made either in deriving and adding residuals, or in deriving the mean. If it is believed to be the latter, or if the value used in deriving the residuals was only an assumed approximate value, *the true mean may be quickly derived by subtracting the mean value of the residuals from the assumed mean used in finding the residuals.*

29. Precision of the Arithmetic Mean. — The degree of confidence to be placed in the most probable value of the unknown is shown by its probable error.

(*a*) *Bessel's Formula.*

If we knew the true value x of the unknown, and consequently the true errors Δ_1, Δ_2 . . ., we should have, as in Art. 20, for the m. s. e. of an observation,

$$\mu^2 = \frac{[\Delta^2]}{n}.$$

But we have only the most probable value x_0 and the residuals $v_1, v_2, \ldots v_n$ instead of the true values $x, \Delta_1, \Delta_2 \ldots \Delta_n$. Now,

$$x_0 - v_1 = M_1 = x - \Delta_1,$$

$$x_0 - v_2 = M_2 = x - \Delta_2, \tag{1}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_0 - v_n = M_n = x - \Delta_n.$$

By addition, remembering that $[v] = 0$,

$$nx_0 = nx - [\Delta]. \tag{2}$$

Substitute for x_0 from equation (2) in equations (1) and

$$\begin{aligned} nv_1 &= (n - 1)\,\Delta_1 - \Delta_2 - \cdots \\ nv_2 &= \quad - \Delta_1 + (n - 1)\,\Delta_2 - \cdots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Squaring,

$$\begin{aligned} n^2v_1^2 &= (n - 1)^2\,\Delta_1^2 + \Delta_2^2 + \cdots - 2(n - 1)\,\Delta_1\Delta_2 - \cdots \\ n^2v_2^2 &= \Delta_1^2 + (n - 1)^2\,\Delta_2^2 + \cdots - 2(n - 1)\,\Delta_1\Delta_2 - \cdots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

By addition, assuming that the double products destroy each other,* positive and negative errors being equally probable,

$$n^2[v^2] = \{(n - 1)^2 + (n - 1)\}[\Delta^2].$$

$$\therefore [v^2] = \frac{n - 1}{n}[\Delta^2]$$

$$= (n - 1)\,\mu^2,$$

and

$$\mu^2 = \frac{[v^2]}{n - 1}, \tag{3}$$

which gives the m. s. e. of an observation.

Now, from Art. 13, 14,

$$\mu_0 = \frac{\mu}{\sqrt{n}}.$$

$$\therefore \mu_0 = \sqrt{\frac{[v^2]}{n\,(n - 1)}},$$

which gives the m. s. e. of the arithmetic mean of n observations of equal precision.

* If the positive and negative errors are equal in number, n, there will be a preponderance of negative products. This is too slight to affect the proof. For the proportion of the excess of the negative products to the total number of products is as n to $2n^2 - n$, or as 1 to $2n - 1$, which decreases as the number of errors $2n$ increases.

30. From the constant relation existing between the m. s. e. and p. e. given in Art. 14, we have for the p. e. of an observation and of the arithmetic mean of n observations respectively,

$$r = \rho\sqrt{2}\,\mu$$

$$= \rho\sqrt{2}\sqrt{\frac{[v^2]}{n-1}}. \tag{1}$$

$$r_0 = \rho\sqrt{2}\sqrt{\frac{[v^2]}{n(n-1)}}, \tag{2}$$

where $\rho\sqrt{2} = 0.6745$ nearly.

31. It is important to note carefully the distinction between residuals and errors. The errors are quantities of which we may never hope to secure the exact values, since they are the differences between the true value and the separate observed values. We cannot secure the true value. We can secure a most probable value which is an approximation to the true value. The residuals are the differences between the most probable value and the separate observed values. The residuals are approximations to the corresponding errors just as the most probable value is an approximation to the true value. The residuals are quantities which may be used in computations. The errors cannot be so used since they are always unknown. The errors appear in the formulas during the process of deriving them, but they necessarily disappear from the formulas before they are in shape to be used by the computer.

32. Peters' Formula. — The m. s. e. and p. e. of a series of observed values may be more rapidly computed from the sum of the errors rather than from the sum of their squares by means of the convenient formula first given by Dr. Peters.*

From the equation,

$$[v^2] = \frac{n-1}{n}[\Delta^2],$$

we have approximately, without regard to sign,

* *Astronomische Nachrichten*, No. 1034.

$$v_1 = \sqrt{\frac{n-1}{n}}\,\Delta_1,$$

$$v_2 = \sqrt{\frac{n-1}{n}}\,\Delta_2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Adding and dividing by n,

$$\frac{[v}{n} = \sqrt{\frac{n-1}{n}}\,\eta,$$

where $[v$ is the sum of the residuals without regard to sign.

But from Art. 16,

$$\mu = \sqrt{\frac{\pi}{2}}\,\eta.$$

$$\therefore \mu = \sqrt{\frac{\pi}{2n(n-1)}}\,[v$$

$$= \frac{1.2533}{\sqrt{n(n-1)}}\,[v \text{ nearly.}$$

For the p. e. of an observation and of the arithmetic mean of n observations we have respectively

$$r = \rho\sqrt{2}\,\mu = \frac{0.8453}{\sqrt{n(n-1)}}\,[v,$$

$$r_0 = \frac{0.8453}{n\sqrt{n-1}}\,[v.$$

33. Collecting the formulas for finding the p. e. of a single observation and of the arithmetic mean of n observations, we have

$$r = 0.6745\sqrt{\frac{[v^2]}{n-1}}, \qquad r = 0.8453\,\frac{[v}{\sqrt{n(n-1)}},$$

$$r_0 = 0.6745\sqrt{\frac{[v^2]}{n(n-1)}}, \qquad r_0 = 0.8453\,\frac{[v}{n\sqrt{n-1}}.$$

To save labor in the numerical work, I have computed tables containing the values of the coefficients of $\sqrt{[v^2]}$ and $[v$ in these

equations for values of n from 2 to 100. (See Appendix, Tables II, III.)*

If Bessel's formula is used, compute first $[v^2]$, then $\sqrt{[v^2]}$ can be taken from a table of squares closely enough. This square-root number multiplied by the number in Table II corresponding to the given value of n gives the p. e. sought. If Peters' formula is used, multiply the sum of the residuals, without regard to sign, by the numbers in Table III corresponding to the argument n.

34. Control of $[v^2]$. — A control is afforded by the derivation of $[v^2]$ from the observed values and the arithmetic mean directly.

We have

$$\begin{aligned} v_1 &= x_0 - M_1, \\ v_2 &= x_0 - M_2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_n &= x_0 - M_n. \end{aligned}$$

Square and add,

$$\begin{aligned} [v^2] &= nx_0^2 - 2\,x_0[M] + [M^2] \\ &= [M^2] - [M]\,x_0; \end{aligned} \tag{1}$$

since $$nx_0 = [M].$$

The values of M^2 may be found from a table of squares or from Crelle's tables, or, if the numbers M are large, an arithmometer, or machine for multiplying and dividing, may be employed with advantage.

35. Approximate Method of Finding the Precision. — A connection between the p. e. of a single observation and the greatest error committed in the series may be established approximately by the aid of the principle proved in Arts. 22–23. There we saw that in a large series the actual errors may be expected to range between zero and 4 or 5 times the p. e. of an observation. If, then, we find from the observations a p. e. of an amount, say, r, we may assert that the greatest actual error is

* First published in the *Analyst*, Des Moines, Ia., May, 1882.

not likely to be more than $5\,r$. The probability of its being as large as this is only about $\frac{1}{1000}$.

The same principle will enable us to estimate roughly the p. e. in a series of observations. A glance at the measured results will show the largest and smallest, and their difference may be taken as the range in the results, and half the difference as the maximum error. Hence, since in an ordinary series of from 25 to 100 observations the maximum error may be expected to be from 3 to 4 times the p. e., we may take *the p. e. to be from $\frac{1}{6}$ to $\frac{1}{8}$ of the range of the errors of observation.*

The probable error so estimated is, however, rather untrustworthy, as it depends upon but two of the residuals instead of all of them. Moreover, these are the two residuals which correspond to the extreme observations, about which there is frequently a reasonable doubt as to whether they should not be rejected.

36. **Ex.** — We shall now apply the preceding formulas to the example in Art. 27 to find the p. e. of the arithmetic mean and of a single observation.

(1) *The p. e. of the arithmetic mean.*

These we may find in two ways:

(*a*) From the sum of the squares of the residuals (Art. 29),

$$\begin{aligned}\mu_0 &= \sqrt{\frac{[v^2]}{n\,(n-1)}}\\ &= \sqrt{\frac{0.2756}{21 \times 20}}\\ &= 0.026,\\ r_0 &= 0.6745 \times 0.026\\ &= 0.017;\end{aligned}$$

or from Table II at once:

$$\begin{aligned}r_0 &= 0.525 \times 0.033\\ &= 0.017.\end{aligned}$$

(*b*) From the sum $[v]$ of the residuals (Art. 32):

The multiplier in Table III corresponding to the number 21 is 0.009.

$$\begin{aligned}\therefore\ r_0 &= 2.02 \times 0.009\\ &= 0.018.\end{aligned}$$

(2) *The p. e. of a single observation.*

From Tables II. and III. directly:

$$r = 0.525 \times 0.151 = 0.079,$$
$$r = 2.02 \quad \times 0.041 = 0.082.$$

Check (α). Nine residuals out of twenty-one are less than the computed p. e. $\pm 0.079''$, whereas, according to theory (Table I), one-half the errors, or $10\frac{1}{2}$ out of 21, should be less than $\pm 0.079''$.

This is the practical way of using the check. We might have arranged the residuals in order of magnitude when the residual 0.09 will be found to occupy the middle place.

Check (β). See Art. 35.

$$\text{Range} = 0.22 + 0.23 = 0.45.$$

$$\therefore r = \frac{0.45}{6} = 0.08.$$

The values found by the different methods agree reasonably well.

37. The Law of Error Tested by Experience. — We shall now test our example and see how closely it conforms to the law of error, and hence be in a better position to judge of how far the law of error itself is applicable in practice. This is the *à posteriori* proof intimated in Art. 6 as necessary for the demonstration of the law.

(1) The number of + residuals is 12, and the number of − residuals is 9.

(2) The sum of the + residuals is 1.03, and the sum of the − residuals is 0.99.

(3) The sum of the squares of the + residuals is 1417, and of the − residuals is 1339.

(4) The p. e. of a single observation is 0.08. To find the number of observations we should expect whose residual errors are not greater than 0.10, we enter Table I with the argument $\dfrac{0.10}{0.08} = 1.25$ and find 0.60. This multiplied by 21 gives 13 as the number of errors to be expected not greater than 0.10. By actual count we find the number observed to be 14.

To find the number to be expected between 0.10 and 0.20 we enter the table with the argument $\dfrac{.20}{.08} = 2.50$ and find 0.91. From this deduct 0.60 and multiply the remainder by 21. This gives 6. The number observed is 5.

The number to be expected over 0.20 is, by theory, 2. The number observed is 2.

The preceding results are collected in the following table:

Limits of Error.	Number of Errors.	
	Theory.	Observation
s. s.		
0.00 to 0.10	13	14
0.10 to 0.20	6	5
over 0.20	2	2

Table I, it will be remembered, is founded on the supposition that the number of observations in a given set is very large. In our example the number is only 21. Perfect accordance between the number of errors given by theory, and the number given by observation is, therefore, not to be expected.

38. Caution as to the Application of the Test of Precision. — In the preceding article we have given several cautions with regard to the strict application of the law of error in practice. We shall now perform a similar service for the test of precision, the probable error. The probable error of an observation, or of the mean of a series of observations, has been defined as a measure of accuracy or in other words of uncertainty. It must be kept clearly in mind that it is a measure only of such uncertainties as are due to accidental errors, and has no necessary relations to systematic or constant errors. If all the errors occurring in the observations are of the accidental class, the probable error is a true measure of accuracy. If systematic or constant errors also occur, these give rise to errors in the result, in addition to those arising from accidental errors, and the p. e., therefore, expresses but a part of the uncertainty of the result. The neglect of this principle has led in many cases to erroneous conclusions, and to faulty methods of observing and computing. These in turn have led to wholesale condemnation of the method of least-squares.

39. The fact that the computed probable error is independent of the constant error in the observations may be shown from the formula from which it is computed.

In the derivation of the p. e. from a series of n observed quantities M_1, M_2, . . . we had the observation equations.

$$x_0 - M_1 = v_1,$$
$$x_0 - M_2 = v_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_0 - M_n = v_n.$$

Also
$$r^2 = \rho \sqrt{2} \frac{[v^2]}{n - 1}.$$

Now, if we suppose each of the observed quantities to be changed by the same amount c, which may be of the nature of a constant error or correction, so that they become $M_1 + c$, $M_2 + c$, . . . the most probable value, instead of being x_0, will be $x_0 + c$. Also since

$$v = (X_0 + c) - (M + c)$$
$$= x_0 - M,$$

the residuals will be the same as before.

Hence r^2 is unchanged, and we see, therefore, that the p. e. makes no allowance for constant errors or corrections to the observed quantity. These are supposed to be eliminated or corrected for before the most probable value and its precision are sought.

In leveling, if the same line is run over in duplicate in the same direction, a good agreement may be expected at the several bench-marks where comparisons are made. The p. e. of observation will consequently be small. If the line is levelled in opposite directions, experience shows that the agreement would not be so good. The p. e. would be larger than before. We might, therefore, hastily conclude that the first work would give the better result. But when we reflect that the main differences arise from such causes as the rising or settling of rods and instruments, the refraction of light, . . . which causes

are less likely to be mutually destructive and more likely to be cumulative if the lines are run in the same direction, it is to be expected that the final result obtained from measurements in opposite directions will be nearer the truth. The conclusion arrived at by trusting to the p. e. alone would be illusory, for the constant and systematic errors in levelling are in general, especially on long lines, much larger than the accidental errors, and the p. e. is simply a measure of the effect of errors of the latter class.

40. Another common misapprehension is the following: From Art. 13 the relation between the p. e of a single observation r and the p. e. of the mean of n observations r_0 is

$$r_0 = \frac{r}{\sqrt{n}}.$$

This formula shows that by repeating the measurement a sufficient number of times we can make the p. e. of the final result as small as we please. Nothing would, therefore, seem to be in the way of our getting an exact result, and that we could do as good work with a rude or imperfect instrument as with a good one by sufficiently increasing the number of observations.

The fallacy lies in the implied supposition that all the errors affecting the observations, are of the accidental class. It is true that the effects of errors of this class will be reduced by increasing the number of observations in the manner indicated above, but experience indicates that in all observations, constant and systematic errors are present as well as accidental errors, being sometimes so small as to be discernible only after the accidental errors have been greatly reduced by many repetitions of the observations, and in other cases so large as to be evident after the first few observations have been taken. The repetition of observations has no effect whatever in eliminating the constant errors, and none in eliminating the systematic errors, unless the conditions under which the observations are taken be by accident or design so changed from time to time as to

reverse the signs of the systematic errors. In general the repetition of observations reduces the error of the result very rapidly at first, while the effects of the accidental errors still predominate over those of other classes, and while each change of one unit in n produces a relatively large change in $\sqrt{n}$. As the observing is continued, the $\sqrt{n}$ changes more and more slowly, and an elimination of the remaining accidental errors from the mean is correspondingly slow. Much more important than this, however, is the fact that a point is, sooner or later, reached in the repetition of observations at which the uneliminated accidental error is smaller than the constant error in the observations and mean. Beyond this point the effect of further observations is simply to reduce the smaller and comparatively unimportant accidental error, and leave the larger serious constant error absolutely unchanged. The effect of indefinitely continuing the observations is to make the combined accidental and constant error, the total error of the result, approach very slowly to the constant error as a limit when the number of observations is infinite.

41. It has been erroneously assumed by many persons that the limit of accuracy for a given instrument is the smallest magnitude that can be seen with it, that "what cannot be seen cannot be measured." The limit of accuracy beyond which one cannot go by increasing the number of observations, is fixed by the constant and systematic errors as indicated above and has no necessary relation to the power of the instrument used.

Three illustrations may be given of measurements of which the errors of the mean are smaller than the smallest quantity which can be seen with the instruments used. With the best theodolites now in use in the Coast and Geodetic Survey, the mean of 16 measures of a direction has a probable error in general from $\frac{1}{4}$ to $\frac{1}{3}$ of a second of arc. The checks which are available show these probable errors to be true measures of the accuracy. These observations are made with a telescope with

which it is impossible under the actual conditions of observation to see a rod or stripe $\frac{1}{12}$ to $\frac{1}{9}$ of an inch wide placed a mile from the instrument, and therefore, subtending an angle of from $\frac{1}{4}$ to $\frac{1}{3}$ of a second.

With the zenith telescope, now being used for the work of determining the variation of latitude under the direction of the International Geodetic Association, the probable error of a single observation has frequently been found to be as small as $\pm 0^s.10$, which is a small fraction of the width of the line used in bisecting the stars and beyond the power of vision of the observer as aided by the telescope. The probable error of the mean result from a night's work is very much smaller than this, $\pm 0^s.04$ or less. The checks obtained from combining the work of different observatories show that these probable errors are true measures of the accuracy, or in other words, that the constant and systematic errors are extremely small.

In the precise level net of the United States, there are thousands of miles of leveling with the new Coast and Geodetic Survey precise level, for which the largest correction expressed in millimeters per kilometer arising from the necessity of closing all circuits, the most severe test of accuracy which can be applied, is $\frac{1}{14}$ of a millimeter per kilometer. The readings in this kind of leveling are made on a direct reading rod by estimating the position of each of three cross-lines in the telescope as seen projected against a centimeter graduation on the rod. Each reading is taken to the nearest millimeter only. It is absolutely impossible to see so small a magnitude as $\frac{1}{14}$ millimeter on the rod through the telescope at the distance at which the rod is ordinarily placed, yet the accidental errors are reduced below this limit for a whole kilometer involving in general 12 to 14 sights.

42. The proposition that one cannot measure smaller magnitudes than can be seen is a dangerous error, for the reason that it is liable to lead to poor habits and poor methods of observation. The student, for example, while being taught to use

the zenith telescope for latitude observations, on being warned that he must be extremely careful not to apply any longitudinal pressure on the head of the micrometer screw, will sometimes experiment for himself by purposely applying such a light pressure while watching the bisection. He may not be able to see any change, becomes skeptical, and thereafter is slovenly in his handling of the instrument. Similarly, the observer with a precise level, on being told that an extremely small amount of unequal heating of the level vial will cause a bubble to travel, and introduce an error into the results, will experiment and convince himself that the movement of this character ordinarily encountered under actual field conditions is on an average smaller than he can see. He may then use a method for years which is radically defective in not guarding against this source of error. Although the motion of the bubble may be too small to be visible, yet the principal error in his work may be due to this cause.

43. As an example of a case in which the constant errors are so large that little is gained in accuracy after the first few observations, the determination of absolute declinations and right ascensions may be cited. With the meridian circle Professor Rogers found the p. e. of a single complete observation in declination to be $\pm\ 0''.36$, and the p. e. of a single complete observation in right ascension for an equatorial star to be $\pm\ 0^s.026$. He says: "If, therefore, the p. e. can be taken as a measure of the accuracy of the observations, there ought to be no difficulty in obtaining from a moderate number of observations the right ascension within $0^s.02$ and the declinations within $0''.2$. Yet it is doubtful, after continuous observations in all parts of the world for more than a century, if there is a single star in the heavens whose absolute co-ordinates are known within these limits." * The explanation is, as intimated, that constant errors are not eliminated by increasing the number of observations. Accidental errors are eliminated by so doing.

* *Proc. Amer. Acad. Sci.*, 1878, p. 174.

44. There is a common idea that if we have a poor set of observations, good results can be derived from them by adjusting them according to the method of least squares, or that, if work has been coarsely done, such an adjustment will bring out results of a higher grade. The method of least squares is a method of computation, not of observation, which serves merely to aid the computer to secure in his computed results the highest grade of accuracy possible from a given series of observations, but it cannot increase the accuracy of observations already taken. The observations fix an absolute limit to attainable accuracy. The computer may approach this limit more or less closely according to his skill, but cannot pass it. A third puzzle in connection with probable error may be mentioned. It may happen that the value obtained of the p. e. is numerically greater than that of the observed quantity itself. It is then a question whether in subsequent investigations we should use the value of the observed quantity as found, or neglect it. This depends on circumstances. It is ever a principle in least squares to make use of all the knowledge on hand of the point at issue. If we have strong *à priori* reasons for expecting the value zero, it would be better to take this value. Thus, if we ran a line of levels between two points on the surface of a lake, we should expect the difference of height to be zero. If the p. e. of the result found were greater than the result itself, it would be allowable in this case to reject the determination. On the other hand, when we have no *à priori* knowledge, as in determinations of stellar parallax, for example, if the p. e. of the value found were in excess of the value itself, as is sometimes the case,* we could do nothing but take the value resulting from the observations, unless, indeed, it came out with a negative sign, and then its untrustworthy character would be evident.

45. **Systematic Error.** — References to systematic error in the preceding articles lead us to notice an example or two of

* See, for example, Newcomb, *Astronomy*, app. vii.

the detection and treatment of this great bugbear of observation.

We suspect the presence of systematic errors in a series of observations from finding that the residuals do not bear the relations to each other that they would if the errors were all of the accidental class, or from other departures of the results from the laws which they would follow if the errors were all accidental.

Sometimes the sources of error are detected without much trouble. Thus, in measuring an angle with a theodolite, if the instrument is placed on a stone pillar firmly embedded in the ground, the range in results, if targets are the signals pointed at, would not usually be over $10''$ in primary work; and on reading to a number of signals in order round the horizon, the final reading on closing the horizon would be nearly the same as the initial reading on the same signal. If, next, the instrument were placed on a wooden post, and readings made to signals in order round the horizon in the same way as before, the final reading might differ from the initial by a large amount. The observations might also show that the longer the time taken in going around, the greater the resulting discrepancy. The natural inference would be that in some way the wooden post had to do with the discrepancy in the results. In an actual case * of this kind, examination showed the change to be most uniform on a day when the sun shone brightly. Measurements were then made at night, using lamps as signals on the distant stations, and the same change was observed, only it was in the opposite direction.

The effect on the value of an angle of this twist of station,

* At U. S. Lake Survey station Brulé, Lake Superior, many observations were taken during both day and night in July, 1871, to determine the rate of twist of center-post on which the theodolite used in measuring angles was placed. The conclusion arrived at was that "during a day of uniform sunshine and clear atmosphere this twist seemed to be quite regular, and at the rate of about *one second of arc per minute of time*, reaching a maximum about 7 P. M. and a minimum about 7 A. M., during the month of July. On partially cloudy days there was no regularity in the twist, being sometimes in one direction and again in the opposite."

assuming it to act uniformly in the same direction during the time of observation, can be eliminated by the method of observation: first, reading to the signals in one direction, and then immediately in the opposite direction, and calling the mean of the difference of the two sets of readings a single value of the angle. So also in azimuth work the mean of the difference of the readings, star to mark, and mark to star, gives a single value free from station-twist.

This mode of procedure is in accordance with the general principle to eliminate a systematic error, when possible, by the method of observation, rather than to compute and apply it.

46. The effort to avoid systematic error causes in general a considerable increase of labor, and sometimes this is very marked. For example, in the micrometric comparison of two line measures belonging to the U. S. Engineers, the results found by different observers showed large discrepancies. The micrometer microscopes used were of low power, with a range of about one mm. between the upper and lower limits of distinct vision. Examination showed that the discrepancies arose mainly from focusing, each observer's results being tolerably constant for his own focus. As the value of a revolution of the micrometer screw entered into the reduction of the comparison work, and as this value was obtained from readings on a space of known value, error of focusing entered from this source. Hence a value of the screw had to be determined from a special set of readings taken at each adjustment, and this value used in reducing the regular observations made with the same focus. Had the microscopes been of high power, it would have been sufficient to determine the value of the screw once for all, since the error arising from change of focus could have been classed as accidental.

In trying to avoid or eliminate systematic error, the observer will, as he gains in experience, take precautions which would at first seem to be almost childish. Good work can only be had at the cost of eternal vigilance.

B. Observed Values of Different Quality.

47. The Most Probable Value: the Weighted Mean. — It has been shown in Art. 12 that if the directly observed values $M_1, M_2, \ldots M_n$ of a quantity are of different quality, the most probable value is found by multiplying each residual error of observation by the reciprocal of its p. e., and making the sum of the squares of the products a minimum; that is, with the usual notation,

$$\frac{v_1^2}{r_1^2} + \frac{v_2^2}{r_2^2} + \cdots + \frac{v_n^2}{r_n^2} = \text{a min.,} \tag{1}$$

or

$$\left(\frac{x_0 - M_1}{r_1}\right)^2 + \left(\frac{x_0 - M_2}{r_2}\right)^2 + \cdots + \left(\frac{x_0 - M_n}{r_n}\right)^2 = \text{a min.} \tag{2}$$

By differentiation and reduction,

$$x_0 = \left[\frac{M}{r^2}\right] \div \left[\frac{1}{r^2}\right]. \tag{3}$$

We have, therefore, the equivalent rule:

If the observed values of a quantity are of different quality, the most probable value is found by multiplying each observed value by the reciprocal of the square of its p. e., and dividing the sum of the products by the sum of the reciprocals.

The form of the expression for x_0 suggests another standpoint from which to consider it. Let $p_1, p_2, \ldots p_n$ be the numerical parts of $\frac{1}{r_1^2}, \frac{1}{r_2^2}, \ldots \frac{1}{r_n^2}$, such that each is of the type

$$p = \frac{(\text{unit of measure})^2}{r^2};$$

then equation (3) may be written,

$$x_0 = \frac{[pM]}{[p]} \tag{4}$$

Also, since $\frac{1}{r_1^2}, \frac{1}{r_2^2}, \ldots \frac{1}{r_n^2}$ are similarly involved in the numerator and denominator of the value of x_0, this value will remain the same if $p_1, p_2, \ldots p_n$ are taken any numbers whatever

in the same proportion to $\frac{1}{r_1^2}, \frac{1}{r_2^2}, \ldots \frac{1}{r_n^2}$; that is, if p_1, p_2, . . . p_n satisfy the relations

$$p_1 = \frac{r^2}{r_1^2}, \quad p_2 = \frac{r^2}{r_2^2}, \quad \ldots, \quad p_n^2 = \frac{r^2}{r_n^2} \tag{5}$$

where r is an arbitrary value of the probable error corresponding to the arbitrarily assumed unit weight. The numbers p_1, p_2, p_3, . . . p_n are called the *weights*, or, better, the *combining weights* of the observed values, and the mean value x_0 is called the *weighted mean.*

The expression $[pM]/[p]$ can now be put into words as follows:

If the observed values of a quantity are of different weights, the most probable value is found by multiplying each observed value by its weight, and dividing the sum of the products by the sum of the weights.

In Arts. 13, 14, as indicated in the expressions for the p. e. of a single observation and of the mean, it was shown that for observations of equal precision, or, in other words, of equal weight, the p. e. of a mean of n observations is to the p. e. of a single observation as 1 is to $\sqrt{n}$. By comparison of these with the expressions

$$p_1 = \frac{r^2}{r_1^2}, \quad p_2 = \frac{r^2}{r_2^2}, \quad \ldots, \quad p_n = \frac{r^2}{r_n^2},$$

it may be seen that the meaning of weight p_1 assigned to an observation is that it has the same degree of accuracy as the mean of p_1 observations of unit weight. In combining observations it is treated accordingly. With this understanding, it is evident that to combine observed values M_1, M_2, M_3, . . . M_n, of which the weights are p_1, p_2, p_3, . . . p_n, one should proceed as if M_1 were the mean of p_1 separate observations of unit weight, M_2 of p_2 observations of unit weight, and so on. The arithmetic mean of the hypothetical $[p]$ observations would evidently be $[pM]/[p]$, and this is precisely the form used.

If the observed values M are numerically large we may lighten the numerical work by finding x the method of Art. 26. Proceeding as there indicated, we have

$$\begin{aligned} x_0 &= X' + \frac{[pl]}{[p]} \\ &= X' + x'' \text{ suppose.} \end{aligned}$$

48. Reduction of Observed Values to a Common Standard. — The principle of the weighted mean is evidently an extension of that of the arithmetic mean, as was pointed out long ago by Cotes, Simpson, and others. It merely amounts to finding a mean of several series of means, the unit of the measure being the same in each. As soon, therefore, as results of different weights are changed into others having a common standard of weight, the rules for combining and finding the precision of observed quantities of the same weight can be applied to weighted quantities.

This change we are enabled to make by means of the relation (5), Art. 47, which may be written,

$$r_1 = \frac{r}{\sqrt{p_1}}, \quad r_2 = \frac{r}{\sqrt{p_2}}, \quad \ldots, \quad r_n = \frac{r}{\sqrt{p_n}}.$$

Now, since $r_1, r_2, \ldots r_n$ are the p. e. of $M_1, M_2, \ldots M_n$, the p. e. of $M_1\sqrt{p_1}, M_2\sqrt{p_2}, \ldots M_n\sqrt{p_n}$ would each be the same quantity r.

Hence, if a series of observed values $M_1, M_2, \ldots M_n$ have the weights $p_1, p_2, \ldots p_n$, they are reduced to the same standard by multiplying by $\sqrt{p_1}, \sqrt{p_2}, \ldots \sqrt{p_n}$ respectively.

For example, given the observation equations,

$$\begin{aligned} x - M_1 &= v_1 \text{ weight } p_1, \\ x - M_2 &= v_2 \text{ weight } p_2, \\ &\ldots\ldots\ldots\ldots \\ x - M_n &= v_n \text{ weight } p_n, \end{aligned}$$

to find the most probable value of x.

Reducing to the same standard of weight, we have the equations,

$$\sqrt{p_1}x - \sqrt{p_1}M_1 = \sqrt{p_1}v_1,$$
$$\sqrt{p_2}x - \sqrt{p_2}M_2 = \sqrt{p_2}v_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\sqrt{p_n}x - \sqrt{p_n}M_n = \sqrt{p_n}v_n,$$

and the most probable value of x is found by making

$$(\sqrt{p_1}v_1)^2 + (\sqrt{p_2}v_2)^2 + \ldots + (\sqrt{p_n}v_n)^2 = \text{a min.};$$

that is, by making

$$(\sqrt{p_1}x - \sqrt{p_1}M_1)^2 + (\sqrt{p_2}x - \sqrt{p_2}M_2)^2 + \cdots + (\sqrt{p_n}x - \sqrt{p_n}M_n)^2 = \text{a min.}$$

Reducing this equation, we find, as before,

$$x_0 = \frac{[pM]}{[p]}.$$

49. Control of the Weighted Mean. — Eq. 4, Art. 47, may be written,

$$[pv] = 0.$$

The error of any assigned value of x_0 is evidently $[pl]/[p]$, in which the l's are residuals corresponding to the assumed value of x_0. If this error is less than one-half unit in a given decimal place, the assumed value is correct to that place.

Ex. — Find the most probable value of the velocity of light from the following determinations by Fizeau and others:

298,000 kil.	$\pm$ 1000 kil.	
298,500 "	$\pm$ 1000 "	
299,990 "	$\pm$ 200 "	
300,100 "	$\pm$ 1000 "	
299,930 "	$\pm$ 100 "	

(*Amer. Jour. Sci.*, vol. xix.)

The weights, being inversely as the squares of the probable errors, are as the numbers 1, 1, 25, 1, 100. (Art. 47.)

To avoid handling such large numbers as the M's, we may assume a value X' for x_0 by inspection, say 299,900, and then proceed as follows:

l	p	pl
+ 1900	1	+ 1900
+ 1400	1	+ 1400
− 90	25	− 2250
− 200	1	− 200
− 30	100	− 3000
	128	− 2150

The correction x'' to the assumed value $\dfrac{-(-2150)}{128} = +17$, and the weighted mean $= 299{,}017$.

[It is much more important in computations to keep the numbers as small as possible than to avoid minus signs. Such a procedure saves time. It is almost impossible to arrange computations so that the computer will not be obliged to watch the signs. He can watch many minus signs as well as a few. In the long run, rapidity depends rather upon the number of significant figures used in the quantities handled.]

50. The Precision of the Weighted Mean. — Since the weighted mean x_0 is the arithmetic mean of $[p]$ observations of the unit of weight, its weight is $[p]$. Hence the p. e. r_0 of x_0 is found from

$$r_0^2 = \frac{r^2}{[p]},$$

where r is the p. e. of an observation of the unit of weight (standard observation).

According to Art. 48, the value of r may be found by writing $\sqrt{p_1}v_1$, $\sqrt{p_2}v_2$, . . . for v_1, v_2, . . . in the formulas derived for observations of the same weight. Hence, substituting in Bessel's and in Peters' formulas, Arts. 29 and 32, we have

$$r = .6745\sqrt{\frac{[pv^2]}{n-1}} \qquad \text{or } r = 0.8453\,\frac{[\sqrt{p}v]}{\sqrt{n(n-1)}},$$

and therefore

$$r_0 = .6745 \sqrt{\frac{[pv^2]}{[p](n-1)}} \quad \text{or} \quad r_0 = 0.8453 \frac{[\sqrt{pv}]}{\sqrt{[p]\,n(n-1)}}.$$

These expressions reduce to those for the arithmetic mean where the observed values are of the same weight by putting $[p] = np$.

Ex. — The linear values found for the space $0.00''$ to $0.05''$ of inch $[ab]$ on the standard steel foot I F. of the G. T. Survey of India were as follows: $0.050027''$, $0.049971''$, $0.050019''$, $0.050079''$, $0.050021''$, $0.050011''$. The numbers of measures in these determinations were 6, 6, 15, 15, 8, 8, respectively.

Taking the numbers of measures as the weights of the respective determinations, required the most probable value of the space and its p. e.

The direct solution presents no difficulty. The value of x_0 may be found as in Ex. Art. 49, and thence the residuals v. The p. e. follows from the formulas of Art. 50.

Assume $X' = 0.049971$.

p	l	pl	v	v^2	pv^2
6	+ .000056	+ .000336	+ .000003	.000000000009	.000000000054
6	+ 0	+ 0	+ 59	3481	20886
15	+ 48	+ 720	+ 11	121	1815
15	+ 108	+ 1620	− 49	2401	36015
8	+ 50	+ 400	+ 9	81	648
8	+ 40	+ 320	+ 19	361	2888
58	. . . .	+ .003396	. . . .	. . . . . .	.000000062306

$$\therefore\ x'' = \frac{[pl]}{[p]} = +0.000059, \quad r = 0.6745 \sqrt{\frac{[pv^2]}{(n-1)}} = 0.000075'',$$

$$\text{and } x_0 = X' + x'' = 0.050030, \ r_0 = 0.6745 \sqrt{\frac{[pv^2]}{(n-1)[p]}} = 0.000010''.$$

Hence, $\qquad x_0 = 0.050030'' \pm 0.000010.''$

51. In the above example an important practical point occurs, and one often overlooked. The p. e. is not computed from the original observations, but from these observations grouped in six sets of means. These means we have treated

as if they were original observations of certain weights. Had the original observations been accessible we should have used them, and would most probably have found a different value of the p. e. from that which we have obtained, some of the facts having been partially concealed by the process of taking the means of the separate groups.

In good work the difference to be expected between the value of the p. e. found from the means and that found from the original observations would be small. Still, whenever there is a choice, the p. e. should always be deduced from the original observations rather than from any combinations of them.

The weighted mean value x_0 would evidently be the same whether computed from the partial means or from the original observations.

Observed Values Multiples of the Unknown.

52. Let the observed values $M_1, M_2, \ldots M_n$ be multiples of the same unknown X; that is, be of the form $a_1X, a_2X, \ldots a_nX$, where $a_1, a_2, \ldots a_n$ are constants given by theory for each observation. The values $\frac{M_1}{a_1}, \frac{M_2}{a_2}, \ldots \frac{M_n}{a_n}$ of X may be regarded as directly observed values of unequal weight. If r is the p. e. of an observation, that is, of $M_1, M_2, \ldots$, then, since the p. e. of $\frac{M_1}{a_1}$ is $\frac{r}{a_1}$, of $\frac{M_2}{a_2}$ is $\frac{r}{a_2}$, ... the weights of these assumed observations are proportional to $a_1^2, a_2^2. \ldots$ Hence, taking the weighted mean,

$$X = \frac{\frac{M_1}{a_1}a_1^2 + \frac{M_2}{a_2}a_2^2 + \cdots + \frac{M_n}{a_n}a_n^2}{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$= \frac{[aM]}{[a^2]}.$$

Also, since $[a^2]$ is the weight of X,

$$r_X^2 = \frac{r^2}{[a^2]}.$$

Ex. — To test the power of the telescope of the great theodolite (3 ft.) of the English Ordnance Survey, and find the p. e. of an observation, a wooden framework was set up 12.462 ft. distant from the theodolite when at station Ben More, Scotland. It was so arranged that when projected against the sky a fine vertical line of light, the breadth of which was regulated by the sliding of a board, was shown to the observer. The breadth of this opening was varied by half-inches from $1\frac{1}{2}$ in. to 6 in. during the observations, which were as follows: *

No. of Observations.	Width.	Side of Opening.	Mean of Microscope Readings.
1	6.0	left right	28.00 37.50
2	5.5	left right	28.50 37.00
3	5.0	left right	29.16 37.16
4	4.5	left right	30.16 36.66
5	4.0	left right	30.50 37.16
6	3.5	left right	31.16 37.00
7	3.0	left right	32.66 36.83
8	2.5	left right	33.50 36.83
9	2.0	left right	33.83 37.00
10	1.5	left right	35.50 37.16

Let X = the most probable value of the angle subtending an opening of 1 inch. Then we have the observation equations,

$$\begin{aligned} 6X - 9.50 &= v_1 & 3.5X - 5.84 &= v_6 \\ 5.5X - 8.50 &= v_2 & 3X - 4.17 &= v_7 \\ 5X - 8.00 &= v_3 & 2.5X - 3.33 &= v_8 \\ 4.5X - 6.50 &= v_4 & 2X - 3.17 &= v_9 \\ 4X - 6.66 &= v_5 & 1.5X - 1.66 &= v_{10} \end{aligned}$$

From the preceding we have for the individual values of X and their weights,

* *Account of the Principal Triangulation*, pp. 54, 55.

$$X = 1.58, \quad \text{weight } 6^2.$$
$$X = 1.55, \quad \text{weight } 5.5^2.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore \text{ weighted mean} = \frac{9.50 \times 6^2 + 8.50 \times 5.5^2 + \cdots}{6^2 + 5.5^2 + \cdots}$$
$$= 1.55,$$

or making the sum of the squares of the residuals v a minimum, that is,

$$(6X - 9.50)^2 + (5.5X - 8.50)^2 + \cdots = \text{a min.},$$

we find by differentiation that

$$X = 1.55,$$

as before.

The practical rule following from either method is the same, and may be stated thus: Multiply each observation equation by the coefficient of X in that equation, and add the products. The resulting equation gives the value of X.

Precision of a Linear Function of Independently Observed Quantities.

53. Suppose that there are n independently observed quantities $M_1, M_2, \ldots$ whose m. s. e. are $\mu_1, \mu_2, \ldots$ respectively, to find the p. e. r of F where

$$F = a_1 M_1 + a_2 M_2 + \cdots + a_n M_n, \tag{1}$$

$a_1, a_2, \ldots a_n$ being constants.

If $\Delta_1, \Delta_2, \ldots$ denote the errors of $M_1, M_2, \ldots$ we shall have the true value T of F by writing $M_1 + \Delta_1, M_2 + \Delta_2, \ldots$ for $M_1, M_2, \ldots$ in the above expression for F; that is,

$$T = a_1 (M_1 + \Delta_1) + a_2 (M_2 + \Delta_2) + \cdots + a_n (M_n + \Delta_n).$$

Call Δ the error of F; then, since $T = F + \Delta$, we have

$$\Delta = a_1 \Delta_1 + a_2 \Delta_2 + \cdots + a_n \Delta_n,$$

and $\therefore$

$$\Delta^2 = a_1^2 \Delta_1^2 + a_2^2 \Delta_2^2 + \cdots + 2\, a_1 a_2 \Delta_1 \Delta_2 + \cdots$$

Let the number of sets of $M_1, M_2, \ldots$ required to find T be n, and suppose Δ^2 summed for all the sets of values of $\Delta_1, \Delta_2, \ldots$ and the mean taken, then attending to Art. 13,

$$\mu_T^2 = a_1^2 \mu_1^2 + a_2^2 \mu_2^2 + \cdots + 2\, a_1 a_2 \frac{[\Delta_1 \Delta_2]}{n} + \ldots \tag{2}$$

In forming all possible values of $\Delta_1\Delta_2$, $\Delta_2\Delta_3$, . . ., the number of values being very large, there will probably be about as many + as − products of each form,* and we therefore assume

$$[\Delta_1\Delta_2] = [\Delta_2\Delta_3] = \cdot\cdot\cdot = 0.$$

Hence

$$\mu_T^2 = [a^2\mu^2], \tag{3}$$

and

$$r_T^2 = [a^2r^2].$$

Ex. 1.—The Keweenaw Base was measured with two measuring tubes placed end to end in succession. Tube 1 was placed in position 967 times, and tube 2. 966 times. Given the p. e. of the length of tube 1 = ± 0.00034″, and of tube 2 = ± 0.00037″, find the p. e. in the length of the line arising from the uncertainties in the length of the tubes.

[p. e. from tube 1 = 967 × 0.00034 = 0.329″
p. e. from tube 2 = 966 × 0.00037 = 0.357″
∴ p. e. of line $= \sqrt{0.329^2 + 0.357^2}$
= 0.485″.]

Ex. 2.—In the Keweenaw Base the p. e. of one measurement of 94 tubes, deduced from the discrepancies of six measurements of these 94 tubes, was found to be 0.03″. Show that the p. e. in the length of the line of 1933 tubes arising from the same causes may be estimated at ± 0.136″.

[p. e. of 1 measurement of 1 tube $= \dfrac{0.03}{\sqrt{94}}$

p. e. of base of 1933 tubes $= \dfrac{0.03}{\sqrt{94}}\sqrt{1933}$
$= \pm\, 0.136.$]

Attention is called to these two problems, from the importance of the principles illustrated. In Ex. 1 the p. e. of a tube was multiplied by the whole number of tubes to find the p. e. of the base from that cause, for the reason that with whatever error the tube is affected, it is cumulative throughout the measurement.

In Ex. 2 the p. e. of one tube is multiplied by the square root of the number of tubes, because each measurement is independent of every other, and the errors are as likely to be in excess as in defect, and, therefore, may be expected to destroy one another in the final result.

* See footnote to Art. 29

54. If the function F whose p. e. is required is not in the linear form, we first reduce it to that form. Thus, if

$$F = f(M_1, M_2, \ldots M_n),$$

the true value T of F will result if we write $M_1 + dM_1$, $M_2 + dM_2$, . . . for M_1, M_2, . . . the differentials representing the errors of these quantities. Then

$$T = f(M + dM, M_2 + dM_2, \ldots).$$

Expanding by Taylor's theorem, and retaining only the first powers of the small quantities dM_1, dM_2, . . ., we have,

$$T = F + \frac{\delta f}{\delta M_1} dM_1 + \frac{\delta f}{\delta M_2} dM_2 + \cdots + \frac{\delta f}{\delta M_n} dM_n,$$

or

$$\text{Error of } F = a_1 dM_1 + a_2 dM_2 + \cdots + a_n dM_n \tag{1}$$

where $$a_1 = \frac{\delta f}{\delta M_1}, \; a_2 = \frac{\delta f}{\delta M_2}, \; \ldots \; a_n = \frac{\delta f}{\delta M_n}.$$

This expression is of the same form as (1), Art. 53. Hence,

$$r_F^2 = a_1^2 r_1^2 + a_2^2 r_2^2 + \cdots + a_n^2 r_n^2 = [a^2 r^2].$$

55. Ex. 1. — If r_1, r_2, are the p. e. of the measured sides AB, BC, of a rectangle $ABCD$, find the p. e. of the area of the rectangle.

[Here $$F = M_1 M_2.$$

$\therefore$ by differentiation, $$dF = M_1 dM_2 + M_2 dM_1,$$
and $$r_F^2 = M_1^2 r_2^2 + M_2^2 r_1^2.]$$

Ex. 2. — The expansions of the steel and zinc bars of tube 1 of the Repsold base apparatus of the U. S. Lake Survey for 1° Fahr. are approximately

$$S = 0.0248 \pm 0.0001. \quad \text{mm.}$$
$$Z = 0.0617 \pm 0.0003. \quad \text{mm.}$$

Show that $$\frac{S}{Z} = \frac{2}{5} \pm \frac{1}{400} \text{ nearly.}$$

[For $$F = \frac{S}{Z}.$$

$$\therefore dF = \frac{1}{Z} dS - \frac{S}{Z^2} dZ,$$

and $$(\text{p. e.})^2 = \frac{1}{Z^2} (0.0001)^2 + \frac{S^2}{Z^4} (0.0003)^2.]$$

Ex. 3. — The base b and the adjacent angles A, C of a triangle ABC are measured. If their p. e. are respectively r_b, r_A, r_C, find the p. e. of the angle B and of the side a.

To find r_B.

We have, $$B = 180 + \epsilon - A - C,$$

where ϵ denotes the spherical excess of the triangle.

Hence, A and C being independent of one another,

$$r_B^2 = r_A^2 + r_C^2.$$

To find r_a, $$a = b\frac{\sin A}{\sin B}.$$

By differentiation,

$$da = \frac{\sin A}{\sin B}db + b\frac{\sin(C - \epsilon)}{\sin^2 B}\sin 1''\, dA + a\cot B \sin 1''\, dC,$$

and therefore,

$$r_a^2 = \frac{\sin^2 A}{\sin^2 B}r_b^2 + \frac{b^2 \sin^2(C - \epsilon)\sin^2 1''}{\sin^4 B}r_A^2 + a^2\cot^2 B\sin^2 1''\, r_C^2.$$

Ex. 4. — Given the base b and the angles A, B of a triangle with p. s. e. r_b, r_A, r_B, respectively, to find the p. e. r_a of the side a.

We have $$a = b\frac{\sin A}{\sin B}. \tag{1}$$

This might be expanded as in the preceding example, but more conveniently as follows:

Take logarithms of both members. Then

$$\log a = \log b + \log \sin A - \log \sin B. \tag{2}$$

(a) By differentiation,

$$da = \frac{a}{b}db + a\cot A\sin 1''\, dA - a\cot B\sin 1''\, dB.$$

Hence, $$r_a^2 = \frac{a^2}{b^2}r_b^2 + a^2\cot^2 A\sin^2 1''\, r_A^2 + a^2\cot^2 B\sin^2 1''\, r_B^2. \tag{3}$$

If, as is usually assumed in practice,

$$r_A = r_B = r \text{ and } r_b = 0,$$

then $$r_a = a\sin 1''\, r\sqrt{\cot^2 A + \cot^2 B}. \tag{4}$$

(b) Using log differences, we have by differentiating (2),

$$\delta_a da = \delta_b db + \delta_A dA - \delta_B dB, \tag{5}$$

where δ_a, δ_b are the differences corresponding to one unit for the numbers a

and b in a table of logarithms, and δ_A, δ_B are the differences for 1″ for the angles A and B in a table of log sines. Hence

$$r_a^2 = \left(\frac{\delta_b}{\delta_a}\right)^2 r_b^2 + \left(\frac{\delta_A}{\delta_a}\right)^2 r_A^2 + \left(\frac{\delta_B}{\delta_a}\right)^2 r_B^2. \tag{6}$$

The two equations (3) and (6) may be used to check one another.

The above formulas are true only when the angles A and B are absolutely independent of each other.

This caution is necessary because some far-reaching fallacies as to what constitute good figures in triangulation have resulted from overlooking it.

Ex. 5. — The following example is given for the sake of showing the form of solution by the method of logarithmic differences.

In the triangulation of Lake Superior there were measured in the triangle Middle, Crebassa, Traverse Id (ABC),

$$\angle A = 57^\circ\ 04'\ 51.4'' \quad r_A = 0.30'',$$
$$\angle B = 67^\circ\ 15'\ 39.2'' \quad r_B = 0.29''.$$

The side Middle-Traverse Id. as computed from the Keweenaw Base is 16894.9 yards. Taking $r_b = 0.05$ yd., find r_a and r_C.

We have $$a = b\frac{\sin A}{\sin B}.$$

$$\therefore \log(a + da) = \log(b + db) + \log\sin(A + dA) - \log\sin(B + dB).$$

Then, the differences being expressed in units of the seventh decimal place,

$$\begin{aligned}
&\log(b + db) &&= 4.2277556 + 257\,db\\
&\log\sin(A + dA) &&= 9.9239892 + \ \ 14\,dA\\
&\text{colog}\sin(B + dB) &&= 0.0351398 - \ \ \ 9\,dB\\
&\therefore \log a + \delta_a\,da &&= 4.1868846 + 257\,db + 14\,dA - 9\,dB,\\
\text{and}\quad &283\,da &&= \quad 257\,db + 14\,dA - 9\,dB,\\
\text{since}\quad &\log a &&= 4.1868846,
\end{aligned}$$

and 283 is the difference δ_a as given in the table.

Hence $$r_a^2 = \left(\frac{257}{283}\right)^2(.05)^2 + \left(\frac{14}{283}\right)^2(.30)^2 + \left(\frac{9}{283}\right)^2(.29)^2$$
$$= 0.0024,$$

and $$r_a = 0.05 \text{ yd.}$$

Also, $$r_C = \sqrt{r_A^2 + r_B^2}$$
$$= \sqrt{(0.29)^2 + (0.30)^2}$$
$$= 0.42''.$$

Miscellaneous Examples.

56. Examples of Probable Error.

Ex. 1. — If in a theodolite read by 2 verniers the p. e. of a reading (mean of vernier readings) is $2''$, show that if it is read by 3 verniers the p. e. of a reading will be a little over $1.5''$, and if read by 4 verniers a little less than $1.5''$, provided the errors are of the accidental class.

Ex. 2. — The p. e. of an angle of a triangle is r; show that the p. e. of the closing error of the triangle is $r\sqrt{3}$, all of the angles being equally well measured.

[Closing error $= 180^\circ - (A + B + C)$.]

Ex. 3. — The length of a measuring bar at the beginning of a measurement was $a \pm r_1$. After x measures had been made, it was $b \pm r_2$. Show that the length of the nth measure, the length being supposed to change uniformly with the distance measured, is

$$a + \frac{n}{x}(b - a) \pm \sqrt{\left(1 - \frac{n}{x}\right)^2 r_1^2 + \frac{n^2}{x^2} r_2^2}.$$

[For if da is the error of a, and db of b, then the error of

$$a + \frac{n}{x}(b - a) \quad \text{is} \quad \left(1 - \frac{n}{x}\right) da + \frac{n}{x}\, db,$$

and the above p. e. follows.

It is a common mistake to write the error in the form $da + \frac{n}{x}(db - da)$, and hence to infer that the p. e. is $\sqrt{r_1^2 + \frac{n^2}{x^2}(r_1^2 + r_2^2)}$.]

Ex. 4. — Prove that the p. e. of the mean of two observations whose difference is d is $0.337\, d$, and the p. e. of each observation is $0.477\, d$.

Ex. 5. — The line Monadnock-Gunstock (94469 m.) was computed from the Massachusetts Base (17326 m.) through the intervening triangulation. The p. e. of the line arising from the triangulation is $\pm$ 0.317 m., and the p. e. of the base is 0.0358 m.; find the total p. e. of the line.

$$\left[\text{p. e.} = \sqrt{\left(\frac{94{,}469}{17{,}326} \times 0.0358\right)^2 + (0.317)^2} = \pm\ 0.372 \text{ m.}\right]$$

Ex. 6. — The Minnesota Point Base reduced to sea-level is

1325×15 ft. bar at $32^\circ + 11.314$ in. $\pm$ 0.421 in.

and 15 ft. bar at $32^\circ = 179.95438$ in. $\pm$ 0.00012 in.

show that the p. e. of the base is $\pm$ 0.450 in.

[p. e. $= \sqrt{(1325 \times 0.00012)^2 + (0.421)^2} = \pm 0.450$ in. We multiply $\pm$ 0.00021 inches by 1325: uncertain which sign it is; but whichever it is, it is constant all the way through.]

Ex. 7. — If the zenith distance ζ of a star is observed n_1 times at upper culmination, and the zenith distance ζ' of the same star is observed n_2 times at lower culmination, show that the m. s. e. of the latitude of the place of observation is

$$\frac{\mu}{2}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}},$$

μ being the m. s. e. of a single observation.

$$[\text{Latitude} = 90^\circ - \tfrac{1}{2}(\zeta + \zeta').]$$

Ex. 8. — Given the telegraphic longitude results,

		h.	m.	s.	s.
	Cambridge west of Greenwich	= 4	44	30.99	± 0.23
	Omaha, west of Cambridge	= 1	39	15.04	± 0 06
	Springfield east of Omaha	=	25	08.69	± 0.11
show that	Springfield west of Greenwich	= 5	58	37.34	± 0.26

$$[\text{p. e.} = \sqrt{.23^2 + .06^2 + .11^2} = 0.26.]$$

Ex. 9. — Given mass of earth + mass of moon $= \dfrac{1}{305{,}879 \pm 2271}$,

and mass of moon $= \dfrac{1}{81.44}$ mass of earth,

prove mass of earth $= \dfrac{1}{309{,}635 \pm 2299}$.

$$\left[\text{For } (305{,}879 \pm 2271) \times \frac{82.44}{81.44} = 309{,}635 \pm 2299.\right]$$

Ex. 10. — In measuring an angle suppose

r_1 = p. e. of a pointing at a signal,
r_2 = p. e. of a reading of the limb of the instrument,
e = error of graduation of the arc read on ;

then, assuming that these result from the only sources of error not eliminated, show if the limb has been changed m times, and n readings taken in each position, that

$$\text{p. e. of angle} = \pm\sqrt{\frac{2(r_1^2 + r_2^2)}{mn} + \frac{e^2}{m}}.$$

For one position of the limb,

$$\text{p. e. of angle} = \pm\sqrt{\frac{2(r_1^2 + r_2^2)}{n} + e^2},$$

as the error of graduation remains constant throughout each set of n readings.

It is important to note that the n-1 readings in each position after the first reading has been taken reduce the effects of pointing and reading errors but not of errors of graduation.

What would the p. e. of an angle have been if each of the mn readings had been taken in a new position?

Ex. 11.—The distance o – 1 mm. on a graduated line-measure is read with a micrometer; show that the p. e. of the mean of two results is equal to the p. e. of a single reading.

[For distance o – 1 mm.

$$= \tfrac{1}{2}\{(\text{first} + \text{second rdg.}) \text{ at o} - (\text{first} + \text{second rdg.}) \text{ at 1 mm.}\}$$

$$\therefore\ (\text{p. e.})^2 = \tfrac{1}{4}\{4\,(\text{p. e.})^2 \text{ of a rdg.}\}]$$

Ex. 12.—In the comparison of a mm. space on two standards placed side by side and read with a micrometer, the p. e. of a single micrometer reading being a, show that the p. e. of the difference of the results of n combined measurements (each being the mean of two measurements) is $\sqrt{\frac{2}{n}}\,a$.

[For p. e. of a reading $= a$.

$\therefore$ p. e. of a combined measurement $= a$,

and p. e. of mean of n combined measurements $= \dfrac{a}{\sqrt{n}}$ · etc.]

Ex. 13.—A theodolite is furnished with n reading microscopes, all of the same precision. A graduation-mark on the limb is read on m times with a single microscope, giving the p. e. of a single reading to be r_1. The telescope is then pointed at an object m times, and the p. e. of the mean of the microscope readings is found to be r_2. Show that the p. e. of a pointing is

$$\sqrt{r_2^2 - \frac{r_1^2}{n}}.$$

[p. e. of reading (mean of verniers) with n microscopes $= \dfrac{r_1}{\sqrt{n}}$.

Total error = error of reading + error of pointing.

$$\therefore r_2^2 = \frac{r_1^2}{n} + (\text{p. e. of ptg.})^2, \text{ etc.}]$$

Ex. 14.—If r_1b_1, r_2b_2 are the p. e. of the base measurements, and $r_3\lambda$ the p. e. of the ratio λ, given by the triangulation, of a base b_2 to a base b_1, show that the p. e. of the discrepancy between the computed and measured values of b_2 is $b_2\sqrt{[r^2]}$.

$$[\text{Discrepancy} = b_2 - b_1\lambda = l \text{ suppose.}$$

$$\therefore\ db_2 - b_1 d\lambda - \lambda db_1 = dl,$$

and

$$r_2^2 b_2^2 + b_1^2(r_3\lambda)^2 + \lambda^2(r_1b_1)^2 = r^2,$$

or

$$b_2^2\,(r_2^2 + r_3^2 + r_1^2) = r^2.]$$

Ex. 15.—At the time t_1 the correction to a chronometer was $a_1^s \pm r_1$, and at the time t_2 it was $a_2^s \pm r_2$; show that the p. e. of the rate of the chronometer is $\dfrac{\sqrt{r_1^2 + r_2^2}}{t_2 - t_1}$ and find the p. e. of the correction to the chronometer at an interpolated time t'.

$$\left[\text{Correction} = a_1 + \frac{a_2 - a_1}{t_2 - t_1}(t' - t_1) \text{ at time } t'.\right.$$

$$\left.\therefore \text{p. e.} = \frac{\sqrt{(t_2 - t')^2 r_1^2 + (t' - t_1)^2 r_2^2}}{(t_2 - t_1)}.\right]$$

Ex. 16. — Given
$$x \cos \alpha = l_1 \pm r_1,$$
$$x \sin \alpha = l_2 \pm r_2,$$

find p. e. of x and of α.

$$\left[dx = \frac{\delta x}{\delta l_1} dl_1 + \frac{\delta x}{\delta l_2} dl_2\right.$$

$$= \frac{l_1}{\sqrt{l_1^2 + l_2^2}} dl_1 + \frac{l_2}{\sqrt{l_1^2 + l_2^2}} dl_2$$

$$\left.\therefore \text{p. e. of } x = \sqrt{\frac{l_1^2 r_1^2 + l_2^2 r_2^2}{l_1^2 + l_2^2}}. \quad \text{Similarly p. e. of } \alpha = \frac{\sqrt{l_2^2 r_1^2 + l_1^2 r_2^2}}{l_1^2\, l_2^2}.\right]$$

Ex. 17. — Given on a line-measure the p. e. of a distance OA measured from O to be r_1, and of OB, also measured from O, to be r_2; find the p. e. of OD when D is the middle point of AB.

$$[OD = \tfrac{1}{2}(OA + OB).$$
$$\therefore r = \tfrac{1}{2}\sqrt{r_1^2 + r_2^2}.$$

If $r_1 = r_2 = 1^\mu$, for example, then $r = .08^\mu$, when μ = one micron.

It may at first sight appear paradoxical that the p. e. of the computed quantity may be smaller than the p. e. of the measured. It is evident, however, that the error of OD is one-half the sum of the errors of OA and OB. If the signs of the errors are alike, the error of OD is *never* greater than the larger of the errors; if the signs are different, it is *always* less.]

Ex. 18. — Given the p. e. of x to be r; find the p. e. of $\log x$.

$$\left[d \log x = \frac{\text{mod.}}{x} dx.\right.$$

$$\left.\therefore \text{p. e.} \log x = \frac{\text{mod.}}{x} r.\right]$$

Ex. 19. — In the measurement of the Massachusetts base line, consisting of 2165 boxes, the p. e. of a box, as derived from comparisons with the standard meter, was $\pm$ 0.0000055 m., the p. e. from instability of microscopes in measuring a box was 0.000127 m., and the p. e. of the base from temperature corrections was 0.0332 m. Show that the p. e. of the base arising from these independent causes combined is 0.0358 m.

$$[\text{p. e.} = \sqrt{(2165 \times 0.0000055 \text{ m.})^2 + (0.000127 \text{ m.} \sqrt{2165})^2 + (0.0332 \text{ m.})^2}$$
$$= \pm 0.0358 \text{ m.}]$$

Ex. 20. — Given the length of the Massachusetts base to be 17326.3763 m. $\pm$ 0.0358 m.; show that the corresponding value of the p. e. of its logarithm is 8.973 in units of the seventh place of decimals.

$$[\log(b \pm 0.0358) = \log b \pm \frac{\text{mod.}}{b}(0.0358).$$

log mod.	9.6377843
log 0.0358	8.5538830
	8.1916673
log b	4.2387077
0.0000008973	3.9529596]

Ex. 21.—The p. e. of the log of a number N in units of the seventh decimal place is $\pm$ 10.6; find the ratio of the p. e. to the number.

$$\left[\log(N + v) = \log N + \frac{\text{mod.}}{N} v,\right.$$

$$r^2_{\log(N+v)} = \left(\frac{\text{mod.}}{N}\right)^2 r^2.$$

$$\therefore \frac{\text{mod.}}{N} r = 10.6 \div 10^7,$$

and

$$\left.\frac{r}{N} = \frac{1}{410,000}.\right]$$

57. Examples of the Weighted Mean.

Ex. 1.—The weights of the independently measured angles BAC, CAD, DAE, are 3, 3, 1 respectively; find weight of the sum-angle BAE. *Ans.* 0.6.

To solve this, write the expression for (p. e.)^2. In general, the weight of the combination of several quantities united by + or − signs, thus,

$$X = X_a - X_b - X_c + \ . \ . \ .$$

is given by the expression

$$\frac{1}{P} = \frac{1}{p_a} + \frac{1}{p_b} + \frac{1}{p_c} + \ . \ . \ .$$

Ex. 2.—If $X = a_1x_1 + a_2x_2 + \ . \ . \ . + a_nx_n$, and $p_1, p_2, \ . \ . \ . \ p_n$ are the weights of $x_1, x_2, \ . \ . \ . \ x_n$, and P the weight of X, show that

$$\frac{1}{P} = \left[\frac{aa}{p}\right].$$

Ex. 3.—Professor Hall found, from observations of the satellites of Mars, that from Deimos, mass of Mars $= \dfrac{1}{3,095,313 \pm 3485}$, and from Phobos, mass of Mars $= \dfrac{1}{3,078,456 \pm 10,104}$, the mass being expressed in the common unit. Show that, taking the weighted mean, we have approximately,

$$\text{mass of Mars} = \frac{1}{3,093,500 \pm 3295}.$$

Ex. 4.—On a graduated bar the space 0 − 1 m. is measured and found

to be 1 m. with a weight 1, and the space 0 – 2 m. is measured and found to be 2 m. with a weight 2; required the value of the space 1 m. – 2 m. and its weight P.

[Space 1 m. – 2 m. = 1 m. It makes no difference what the weights are so far as the value of the space is concerned.

To find P, (1 m. – 2 m.) = (0 – 2 m.) – (0 – 1 m.).

$$\therefore \frac{1}{P} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \text{ and } P = \frac{2}{3}.]$$

Ex. 5. — Given the weight of $x = p$, show that

$$\text{weight of } \log x = \frac{x^2}{(\text{mod.})^2} p.$$

Ex. 6. — If $x = \frac{y}{c}$ and the weight of y is p, then

$$\text{weight of } x = c^2 p.$$

Ex. 7. — Given the results for difference of longitude, Washington and Key West,

		m. s. s.
1873,	Dec. 24,	19 01.42 ± 0.044
	Dec. 26,	1.37 ± .037
	Dec. 30,	1.38 ± .036
	Dec. 31,	1.45 ± .036
1874,	Jan. 9,	1.60 ± .046
	Jan. 10,	1.55 ± .045
	Jan. 11,	19 01.57 ± 0.047

show that

		m. s. s.
Weighted mean	=	19 01.460 ± 0.016,
Weighted mean of first four nights	=	19 01.404 ± 0.019,
Weighted mean of last three nights	=	19 01.573 ± 0.027,

and from the last two results check the first.

Ex. 8. — In the triangulation connecting the Kent Id. Base, Md., and the Craney Id. Base, Va., the length of the line of junction computed from

		m. m.
Kent Id. Base	=	26758.432 ± 0.38,
Craney Id. Base	=	26758.176 ± 0.43.

Show that

		m. m.
(1) Discrepancy of computed values	=	0.256 ± 0.57.
(2) Most prob. length of junction line	=	26758.32 ± 0.28.

Ex. 9. — In latitude work with the zenith telescope, if n north stars are combined with s south stars, giving ns pairs, to find the weight of the combination, that of an ordinary pair, one north and one south, being unity.

[Let r = the p. e. of an observation of one north star or of one south star.

Then, as though combining the mean of n north stars with the mean of s south stars, the wt. p of the combination is

$$\frac{1}{p} = \frac{r^2}{n} + \frac{r^2}{s}.$$

But

$$1 = \frac{r^2}{1} + \frac{r^2}{1}.$$

$$\therefore p = \frac{2\,ns}{n+s}.\Bigg]$$

"The combination of more than two stars gave some trouble. In one case there were 3 north and 4 south, which would give 12 pairs, but with a weight of $2\dfrac{3 \times 4}{3+4}$ only. In this and all similar cases I treated the whole combination as one pair; that is, I inserted in the blank provided the half-sum of the mean of the declinations of north stars and of the mean of the declinations of south stars, and gave the result a higher weight. This is the only logical method." (Safford, *Report, Chief of Engineers U. S. A.*, 1879, p. 1987.)

For a series of examples by Airy on the weights to be given to the separate results for terrestrial longitude determined by the observations of transits of the moon and fixed stars, see *Mem. Roy. Astron. Soc.*, vol. xix.

Ex. 10. — If a close zenith star is observed with a zenith telescope first as a north star, and immediately after as a south star, show that the weight of the resulting latitude is less than that found from observing an ordinary pair.

Ex. 11. — In the triangulation of Lake Ontario the angle Walworth-Palmyra-Sodus was measured as follows:

In 1875, with theodolite P. and M. No. 1,

$$74^\circ\ 25'\ 05.429'',\ \pm 0.29'',\ \text{mean of 16 results.}$$

In 1877, with theodolite T. and S. No. 3,

$$74^\circ\ 25'\ 04.611''\ \pm 0.22'',\ \text{mean of 24 results;}$$

required the most probable value of the angle and its probable error

The weights are in the ratio $\dfrac{(0.22)^2}{(0.29)^2}$.

NOTE. — If, instead of being two measurements of the same angle, the above were the measurements of two angles side by side, then

$$\text{total angle} = 148^\circ\ 50'\ 10.040'',$$

because, no matter how much better one is measured than the other, we can do nothing but take the sum of the two values.

Ex. 12. — An angle is measured n times with a repeating theodolite, and also n times with a non-repeating theodolite, the precision of a single reading and of a single pointing being the same in both cases; compare the weights of the results.

[r_1, r_2 the p. e. of a single pointing and of a single reading.

With a non-repeating theodolite each measurement of the angle contains

$$(\text{pointing} + \text{reading}) - (\text{pointing} + \text{reading})$$

$$\therefore \text{(p. e.)}^2 \text{ of one measurement} = 2\,r_1^2 + 2\,r_2^2,$$

and
$$\text{(p. e.)}^2 \text{ of mean of } n \text{ measurements} = \frac{1}{n}(2\,r_1^2 + 2\,r_2^2).$$

With a repeating theodolite the successive measurements of the angle are

$$\begin{array}{r}(\text{pointing} + \text{reading}) - \text{pointing}\\ \text{pointing} - \text{pointing}\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \text{pointing} - (\text{pointing} + \text{reading}).\end{array}$$

$$\therefore \text{(p. e.)}^2 \text{ of } n \text{ times the angle} = 2\,nr_1^2 + 2\,r_2^2,$$

and
$$\text{(p. e.)}^2 \text{ of the angle} = \frac{1}{n^2}(2\,nr_1^2 + 2\,r_2^2).$$

If, then, p_1, p_2 denote the weights of an angle resulting from n reiterations or from n repetitions,

$$p_1 : p_2 = 1 + \frac{r_2^2}{nr_1^2} : 1 + \frac{r_2^2}{r_1^2}$$

and hence it would seem that the method of repetition is to be preferred to the method of reiteration. This advantage is so much less, the smaller $\frac{r_2^2}{r_1^2}$ is; that is, the more the precision of the circle reading increases in proportion to the precision of the pointing.

This result is contradicted by experience.

The result obtained is true on the hypothesis that only accidental errors enter. We have assumed a perfect instrument. But the instrument-maker cannot give what the geometer demands. From various mechanical reasons the systematic error in a repeating theodolite increases with the number of observations, whereas in the reiterating theodolite it disappears. This systematic error, in whatever way it arises, causes the trouble. It is so difficult to eliminate the systematic error in observations with a repeating theodolite, that in spite of its advantages over the direction theodolite in so far as accidental errors are concerned, it is still an open question which type of instrument will give greater accuracy. In the Coast and Geodetic Survey both types are in use. The direction theodolite is, however, used much more than the repeater on primary triangulation.

Of the Weighting of Observations.

58. When the sources of error are of such kinds that, so far as we know, they cannot be separated, the p. e. and consequent weight are found as described in the preceding sections. The weight has been defined as a number representing the relative

goodness of an observation, or of a result computed from observations, and as a number inversely proportional to the square of the p. e. of an observation or computed result. In assigning or in computing weights it must be kept continually in mind that the intention is to make weights inversely proportional to the squares of the p. e. arising from all sources, and that no avenue of information should be neglected. To assign weights which are simply inversely proportional to the computed p. e.'s which may happen to be available without careful examination as to their trustworthiness is not good practice, nor is it in accordance with the complete theory of the method of least squares. The object of the following paragraphs is to indicate some of the precautions which must be taken in assigning weights.

For example, two separate determinations of a millimeter space, made in the same way, gave

$$1000.1^{\mu} \pm 0.40^{\mu}, \text{ mean of 20 readings,}$$
$$1000.3 \pm 0.33, \text{ mean of 30 readings.}$$

To find the weighted mean of these two sets of measurements, we may proceed in two ways. The number of results in the first measurement is 20, and the number in the second is 30. Hence, taking the weights proportional to the number of results, the mean

$$= 1000 + \frac{20 \times 0.1 + 30 \times .3}{20 + 30} = 1000.22.$$

Again, since the p. e. of the measurements are 0.40 and 0.33, their weights are as $\frac{1}{40^2}$ to $\frac{1}{33^2}$, that is, as 1089 to 1600, and the resulting weighted mean is 1000.22, agreeing with the other computed value to the last decimal place retained.

59. In this example the two methods of computation give exactly the same result. It is not always so. Some "run of luck," or balancing of errors, or constant conditions, might have made the observations of one set fall very closely together, in which case the weight as computed from the p. e. would have

been very large, while in the other set varying conditions might have caused large ranges and the computed weight would have been small. In such a case the two means might differ considerably. It would then be desirable to study carefully the question as to which method of weighting is probably the better. In such a study it must be kept in mind that the computed probable errors derived from the observations are subject to uncertainty for that reason, and are peculiarly subject to being largely increased by one or a few large residuals. If two series of observations made under similar conditions by the same observer gave computed probable errors which differed widely, it would be an open question whether the actual p. e. of the two series differed widely or whether the difference was simply apparent and due to each of the computed p. e.'s being based on too small a number of observations to be a close approximation to the truth. The computer must use his judgment in deciding this question and assign the weights accordingly. Thus, in the second set of observations above the first three results were 999.8, 999.8, 999.8. The p. e. computed from these values would be zero and the consequent weight infinite. But no one will doubt that these observations are subject to some error, and that the weights assigned to them should be finite and smaller than the weight assigned to the mean of the thirty observations.

60. An Approximate Method of Weighting.—A long-continued series of observations will show the kind of work an instrument is capable of doing under favorable conditions; and if work is done only when the conditions are favorable, the p. e. derived from a certain number of results will generally fall within limits that can be assigned *à priori*. For example, with the Lake Survey primary theodolites, which read to single seconds, the tenths being estimated, the work of several seasons showed that the p. e. of the mean of from 16 to 20 results of the value of a horizontal angle, each result being the mean of a reading with telescope direct and of a reading with telescope reverse, need not be expected to be greater than 0.3″. If, therefore, after having

measured a series of angles in a triangulation net with these instruments, the p. e. all fell within $\pm$ 0.3$''$, it was considered sufficiently accurate to assign to each angle the same weight.

The objection to this is that "an instrument which has a large periodic error may, if properly used, give as good results as if it had none; but the discrepancies between its combined results for an angle and their mean may be large, thus giving an apparently large probable error to the mean. Moreover, a given number of results over short lines, or lines over which the distant signals are habitually steady when seen in the telescope, will give a resulting value for the angle of much greater weight than the same number of combined results between two stations which are habitually unsteady." *

The same method of weighting was employed by the Northern Boundary Commission in their latitude work. "The standard number of observations [for a latitude determination] was finally fixed at about 60, it being found that with the 32-in. instrument 60 observations would give a mean result of which the p. e. would be about 4 feet."† This method of weighting is based upon the idea that a closer approximation to the truth is obtained in these cases by assuming that the actual p. e.'s are all equal than by accepting their separate computed values as representing the facts accurately. In other words, such a procedure is equivalent to assuming that the principal cause of variation of the separate computed p. e. from a mean value is the accidental grouping of unusually large or unusually small residuals rather than a real variation in accuracy between the different series.

61. Weighting when Constant Error is Present. — The preceding leads us to the case where the error of observation can be separated into two parts, one of which is due to accidental causes, and the other to causes which are constant throughout the observations. The total error e would, therefore, be of the form

$$e = l_1 + l_2.$$

* *Professional Papers of the Corps of Engineers U. S. A.*, No. 24, p. 354.

† *Report, Survey of the Northern Boundary*, p. 86.

This case has been discussed already in general terms in Art. 40 in explaining the well-known fact that an increase in the number of observations with a given instrument does not lead to a corresponding increase of accuracy in the result obtained.

Let

r_1 = the p. e. of the observation arising from the accidental causes,

r_2 = the error peculiar to the observation arising from the constant causes.

Then r_1, r_2, being independent, and being as likely to have opposite signs as the same sign, the total p. e., r of observation may be assumed (Art. 53),

$$r^2 = r_1^2 + r_2^2.$$

If n observations have been made, we shall have for the p. e. r_0 of their mean, since r_2 is constant,

$$r_0^2 = \frac{r_1^2}{n} + r_2^2.$$

It is evident that when n is large, r_2^2 becomes the important term, and that in any case the value of r_0 and consequent weight can be but little improved by increasing the number of observations.

62. For the purpose of finding the value of the p. e. arising from the constant sources of error, a special series of observations is, in general, necessary. After this series has been made, the value of r_2 found from it can be applied in the determination of the value of r_0 in any other series made under like conditions.

For illustration let us consider a latitude determination with the zenith telescope. The zenith-distance, ζ, of each star being observed, the half-difference of zenith-distances for each pair may be computed, and each of these computed values may be considered an observed value. The values of the declinations δ are taken from a catalogue of stars. The errors of δ are, therefore, independent of those of ζ, and are constant for the same pair of stars. The latitude ϕ from one pair is given by

$$\phi = \tfrac{1}{2}(\delta' + \delta) + \tfrac{1}{2}(\zeta' - \zeta).$$

Let

$r_\zeta =$ the p. e. of $\frac{1}{2}(\zeta' - \zeta)$ for one observation of one pair,
$r_\delta =$ the p. e. of $\frac{1}{2}(\delta' + \delta)$ for this pair,
$r_\phi =$ the p. e. of the resulting latitude ϕ from one pair,

then for a single observation of this pair,

$$r_\phi^2 = r_\delta^2 + r_\zeta^2,$$

and for n observations of this pair,

$$r_\phi^2 = r_\delta^2 + \frac{r_\zeta^2}{n}.$$

The quantity r_ζ will be found from repeated observations of the same pair of stars, as the error in declination will not influence the result. A better value will, of course, be obtained from several pairs than from a single pair. Let, then, many pairs of stars be observed night after night for a considerable period. Collect into groups the latitudes resulting from the observed values of each separate pair. Let $n_1, n_2, \ldots n_m$ be the number of results in the several groups, the number in any group being at least two. Form the residuals for each group and compute the p. e. in the usual way. We have :

No. of Night.	First Pair.		Second Pair.		. . .
	Results.	v	Results.	v	. . .
1	ϕ_1'	v_1'	ϕ_2'	v_2'	. . .
2	ϕ_1''	v_1''	ϕ_2''	v_2''	. . .
3	ϕ_1'''	v_1'''	ϕ_2'''	v_2'''	. . .
. .	. .	. .	. . .	. . .	. . .
Means	ϕ_1		ϕ_2		. . .

Now, assuming that the p. e. of observation of each pair is the same,

$$r_\zeta^2 = \frac{[v_1^2]}{n_1 - 1},$$

$$r_\zeta^2 = \frac{[v_2^2]}{n_2 - 1}.$$

If, then, n is the total number of results, and m the number of groups, by adding the above equations there results

$$r_\zeta^2 = \frac{[v^2]}{n - m}.$$

In finding r_ϕ we assume that though errors of declination are constant for each star, still for a latitude found from many pairs in the same catalogue these errors may be regarded as accidental. Let, then, many different pairs of stars be observed on each of n nights at m' places, no star being observed at more than one place. Collect the means of the single results of each separate pair, and form the residuals v' for each place, taking the differences between these means considered as single results and their mean for that place. Then, reasoning as above, the p. e. of a latitude resulting from n observations on a single pair of stars is

$$r_\phi = 0.6745\sqrt{\frac{[v'^2]}{n' - m'}},$$

where n' is the number of different pairs of stars observed, and m' is the number of places occupied.

Now, r_δ is found from

$$r_\delta^2 = r_\phi^2 - \frac{r_\zeta^2}{n},$$

and is, therefore, known for the star catalogue used. This value may be taken in future work in finding r_ϕ from

$$r_\phi^2 = r_\delta^2 + \frac{r_\zeta^2}{n},$$

and the consequent combining weight of ϕ will be as

$$\frac{1}{r_\phi^2}.$$

63. An example of a similar kind is afforded in finding the weights of the angles measured with a theodolite in a triangulation where more rigid values are required than would be found by Art. 60. The actual error of a measured value of an angle arises from two main sources, errors of graduation and errors of observation. The former are constant for each part of the limb read on, and correspond to the declination errors above, while the latter are incapable of classification, and are, therefore, assumed to be accidental. The periodic errors of graduation are supposed to have been eliminated by proper shiftings of the circle. The resultant p. e. r of a single measurement is found from

$$r^2 = r_1^2 + r_2^2,$$

and the p. e. r_0 of the mean of n measurements made on the same part of the limb from

$$r_0^2 = r_1^2 + \frac{r_2^2}{n},$$

where r_1, r_2 are the p. e. of graduation and observation respectively. The method of treating this problem is quite similar to that of the preceding: r_2 is found by reading the same graduation-mark on the limb many times, and r_0 by reading the angle between two fixed signals many times, the limb being changed after each reading. Thence r_1 is known for the instrument in question, and the combining weights of angles measured with this instrument are at once found.

64. The foregoing leads to another important practical point in the measurement of angles. If the weight of a single observation is unity, then the weight of the mean of n observations made with the limb in one position is

$$p = \frac{r_1^2 + r_2^2}{r_1^2 + \frac{r_2^2}{n}}$$

For certain instruments, experience has shown that we may safely assume

$$r_1 = r_2.$$

and therefore it follows that for these instruments

$$p = \frac{2n}{n+1}.$$

Hence, in using these instruments, no matter how many observations we make in one position of the limb, we never reach the precision of the mean of two observations made with the limb in different positions.

It is evident that to secure the maximum efficiency in the elimination of error, the limb of a direction instrument should be shifted after each reading of an angle. The objection ordinarily urged against such a procedure is that it fails to furnish a sufficient guard against mistakes in reading. The present practice of the Coast and Geodetic Survey is to consider that a pair of readings on each signal, one with the telescope in the direct position, and the other with it in the reverse position, together constitute one observation, and to shift the position of the limb before the next observation. A comparison of the direct and reverse readings furnishes a rough method of detecting mistakes in reading.

65. Assignment of Weight Arbitrarily. — So far we have deduced the combining weights from the observed values themselves, or from them in connection with a special series of observations. But this may not always be the best way of finding the weights. The observations may not be our only source of information, and, indeed, not the most reliable source. If, for example, some phenomenon has been observed by many persons in different parts of the country, and the observations are sent to one place for comparison and reduction, it would not be proper for the computer to deduce a weight for each series from the observations themselves independent of other sources of information he might have. Some of the most inexperienced observers with the poorest instruments might have apparently better results than the most experienced with good instruments. In such a case the computer must exercise his own judgment in classing the observations. He should consider the experience of

the observer, his previous record for accurate work, the kind of instrument used, the conditions, and the observer's record of what he saw — whether it is clear and precise or hazy in its statements. An arbitrary scale of weights may then be constructed, and to each set of observations be assigned a weight from this scale according to the computer's estimate of its value. No two computers would be likely to assign precisely the same weights, but if done by one of experience and good judgment, the result obtained from weighting in this way will undoubtedly be of more value than that found by the strict application of the formulas of least squares.

The point is simply this. The class of observations considered may be expected to contain systematic errors which cannot be determined, and is therefore not capable of being treated by the method of least squares. As we have no direct means of eliminating this kind of error, we must do so indirectly as best we can, and that is what the system of weighting mentioned seeks to accomplish.

An example will be found in the discussion of the *Telescopic Observations of the Transit of Mercury*, May 5–6, 1878, Washington, 1879, where, of 109 observations sent in, to only 18 was the highest weight assigned. Professor Eastman, under whose direction they were reduced, says: " . . . Several instances may be found where small weight is given to observations that apparently agree well with those to which the highest weight is assigned, but in most cases the observer's remarks indicate the uncertain character of the observation."

66. Combination of Good and Inferior Work. — It is strictly in accordance with the idea of weight that if we have two results of very different degrees of accuracy, a result better on the whole than either may be found by combining both with their proper weights. But the proper weights may be difficult to find. On this account it depends on circumstances whether it is advisable to reduce a set of observations poorly made, in order to combine them with a well-made set. If the quantity is

available for observing again, it might not cost any more to do this than to reduce the poor observations. Even if it did, the result would be more satisfactory. The committee of the Royal Society of England which was appointed to examine Col. Lambton's geodetic work in India reported that "Col. Lambton's surveys, though executed with the greatest care and ability, were carried on under serious difficulties, and at a time when instrumental appliances were far less complete than at present. There is no doubt that at the present time the surveys admit of being improved in every part. The standards of length are better ascertained than formerly, and all uncertainty on the unit of measure may be removed. The base-measuring apparatus can be improved. The instruments for horizontal angles used by Col. Lambton were inferior to those now in use. . . . The committee express the strong hope that the whole of Col. Lambton's survey may be repeated with the best modern appliances." *

67. The Weight a Function of our Knowledge. — If a quantity is not available for observing again, as, for example, some transient phenomenon, all of the material on hand must be used, and the best weights possible assigned to the separate values in order to combine them. The point is, that where systematic or constant error has not been eliminated, the weight to be assigned is a function of the state of our knowledge — is, in fact, a matter of individual judgment.

This is brought out very fully in the methods used in combining the older star catalogues with the more modern ones. Thus, Safford (*Catalogue of Mean Declinations of 2018 Stars*, Washington, 1879) says: "In computing positions I have generally employed Argelander's rule, giving to a modern determination from

1 observation a weight $\frac{1}{2}$,
2 observations a weight $\frac{3}{4}$,
3 to 8 observations a weight 1,
9 or more observations a weight $1\frac{1}{2}$ or 2.

* *G. T. Survey of India*, vol. ii. p. 70.

Argelander generally gives Piazzi a weight equal to unity; the value $\frac{1}{2}$ is much nearer the truth; in general he assigns rather a larger relative weight to the older and poorer observations than they deserve. But this is mostly compensated for by the number of determinations."

The weight of a quantity being a function of our knowledge may have assigned to it a certain value at one time and another value at another time when our knowledge of it has increased. Thus, in the Fond du Lac (Wis.) base of the Lake Survey, measured in 1872 with the Bache-Würdemann compensating apparatus, a portion was measured seven times. The results differed widely, far beyond what was expected with the apparatus. No reason could be assigned at the time for the discordances. At this stage, then, one would have been justified in assigning a small weight to the value of the base.

The Keweenaw base was next measured with the same apparatus, and the same trouble came in. Next the Sandy Creek base and then the Buffalo base were measured. During all this time (four years) material had been accumulating for the explanation of the behavior of the apparatus. When the law of its behavior was discovered, it was found that good work not only could be done but had been done with it.

Hence the systematic error being got rid of, one would be justified in increasing the weights of the bases measured with this apparatus in comparison with bases measured with an apparatus of a different kind. Had the later work not been done, the Fond du Lac base would still have had assigned to it the low weight.

Take another instance. Sir G. B. Airy, in 1847, says of the Mason and Dixon arc (*Encyc. Metrop.*, p. 209): "The results of this measure must, we think, be received as equal in authority to those of any other measure." This may have been true when written; but Mr. Schott, in 1877, in his note on the determination of the figure of the earth from American sources, says of this same arc (*U. S. C. S. Report*, 1877, p. 95): "It is,

therefore, only owing to the increased perfection of instrumental means and methods that we now dismiss from further consideration the first measured North American arc, which, moreover, is now superseded by the present measures."

As a third illustration we may consider the weights to be assigned to a system of differences of longitudes in which the connections of the stations occupied are interlaced as in a triangulation net, and the whole system is to be adjusted so as to remove existing contradictions.

If the longitude work has been carried out on one plan, with instruments and observers of about the same quality, then the m. s. e. of each determination may be computed from the measures of the separate nights, and in the adjustment the weights may be taken inversely as the squares of these m. s. e.

But if this has not been done, if in the older work instruments, observers, and methods were poorer than later and the two have to be combined in the adjustment, the computer must estimate as best he can their relative weights. Thus, in a system in Germany, France, and Austria reduced by Dr. Albrecht * the observations were made between the years 1863 and 1876. The methods of observation had been much improved in this interval. In assigning the relative weights, a scale of weights was first formed from a consideration of all the knowledge on hand, taking the march of improvement from year to year into account, and the separate determinations placed in one or other of these classes. Thus, for example,

Weight 1, — No change of observers; few observations; non-adjustment of electric current;

Weight 2, — No change of observers; usual variety of observations; non-adjustment of electric current;

Weight 3, — Change of observers; usual variety of observations; non-adjustment of electric current,

and so on.

Similarly Dr. Bruhns in *Verhandlungen der europäischen*

* *Astronomische Nachrichten*, 2132.

Gradmessung, 1880. See also *Coast Survey Report*, 1880, Appendix 6; 1897, Appendix 2.

68. General Remarks. — The subject of the weighting of observations is confessedly a difficult one. In general it may be affirmed that the less experienced a computer is, the more closely he will adhere to the rigorous formulas without considering whether systematic errors enter or not. As he adds to his experience he will consider outside evidence as well as the evidence afforded by the observations themselves. This will be specially true if he has any practical knowledge of how observations are made. Indeed, it is doubtful if a computer can apply the principles of least squares properly unless he is at least an average observer.

Of the Rejection of Observations.

69. There is nothing in the whole theory of errors more perplexing than the question of what shall be done with an observation of a series which differs widely from the others. In making a series of observations the observer is given full power. He can vary the arrangements, choose his own time for working; he can do anything, in fact, that in his best judgment will tend to give the best value of the observed quantity. But when he has finished observing and goes to computing, has he the same power? Can he alter, reject, manipulate in such a way as in his best judgment will give a result of maximum probability? As observer he was supreme; as computer is he supreme, or only in leading-strings? Various answers may be given to this question, as we look at it from one point of view or another. When observations are made by one man as an expert observer, and reduced by another as an expert computer, the judgment of each should be authoritative in his own province. The observer's statements of fact as to the conditions under which the observations were made must be accepted. His statements of opinion as to the effects of such conditions on the accuracy of the observations must be given great weight, and only set

aside when a considerable mass of carefully considered evidence indicates that the opinion is not correct.

On the other hand, in judging as to the accuracy of a particular observation or group of observations by the residuals alone, the judgment of the computer is authoritative rather than that of the observer. He has in general a much wider acquaintance than the observer with the law of distribution of error, as he deals during his regular routine with the observations of many different men made under many conditions and on different kinds of work. The observer is apt to form his opinion of a given observation, in so far as he judges it by the corresponding residual, by comparing it with the preceding observations in the same series. The computer bases his judgment on all the observations, preceding and following.

70. There is one respect in which experience shows that the opinion of the observer is frequently erroneous. As he sees the conditions under which observations are made, vary from those extremely favorable to accurate measurement to those extremely unfavorable, he more or less explicitly assigns to the observations varying weights. The range of weights assigned is as a rule much too large. If he believes that the observations made under the most unfavorable conditions should be given $\frac{1}{4}$ as much weight as those under the best conditions, the chances are that the ratio of the true weights is more nearly $\frac{1}{2}$ to 1. In other words, the observer's best observations are usually poorer than he believes them to be, and his poorest better. He is misled by his feelings, and estimates the accuracy of the observations by the difficulty of securing them rather than by a careful systematic study of them in the light of all available evidence. The observer is therefore in general too apt to reject observations made under difficult conditions or to decline to observe under such conditions.

71. In the hypothetical case on which the exponential law of error was founded, there were no discontinuous observations taken into account. There we contemplated not only observa-

tions made with the best instruments and by the most experienced observers, but observations of all grades, from this highest grade down to those made with the poorest instruments and by the most ignorant and careless observers conceivable. It is only in this way that errors continuous all the way from $+\infty$ to $-\infty$ could arise. In the cases occurring in ordinary work we confine our attention to one section of the observations only — that made with the good instrument and by the skillful observer. This, to be sure, is the most important, and, as shown in Art. 22, the result following from it differs ordinarily but little from that found in the ideal case. But we are naturally confronted with difficulty when we try to deal with a very incomplete series. Extra assumptions must be made, and it is not to be wondered at that no solution yet offered is regarded as entirely satisfactory.

When it is proposed to reject a certain observation, it should be kept clearly in mind that the only justification for rejection is that by so doing, (1) the effect of a blunder may be eliminated from the final result, (2) or the effect of some error, from an unusual source, of much greater magnitude than the errors affecting the other observations, may be eliminated. The essential difficulty in deciding what to reject is encountered in deciding whether a given large residual is due to either of the causes indicated, or is merely due to the accidental agreement in sign of many small errors from various sources, and is in conformity with the law of error. If the residuals are in strict accordance with the law of error, there will be a few which stand out rather far beyond the general range. (See table 1.) If the observations corresponding to these are rejected, the final result is in general reduced in accuracy rather than increased. Various criterions for rejection have been devised and used, some of them being rather complicated as to their theoretical basis and tedious in application. The following criterion commends itself as being simple, quick of application, and upon a sufficiently good theoretical basis. It is recommended for general use.

Reject each observation for which the residual exceeds five times the probable error of a single observation. Examine each observation for which the residual exceeds $3\frac{1}{2}$ times the probable error of a single observation, and reject it if any of the conditions under which the observation was made were such as to produce any lack of confidence.

The theoretical basis of the rule is evident from an inspection of table 1. If the residuals follow the law of error, but one residual in 55 should exceed $3\frac{1}{2}$ times the probable error of a single observation, and but 1 in 1000 should exceed 5 times that value. The presumption is strong, therefore, that rejections made under the rule are justified in most cases. A few observations will be rejected by this rule which should be retained, but only a few, and therefore little damage will be done in any case.

The probable error for use in the above rule should, of course, be computed from all the observations not rejected up to the time of the proposed application.

72. Rejected observations should be left in the record and computation and in the publication, being simply marked REJECTED. The facts then appear for the inspection of those who follow, and may serve as a basis for independent conclusions.

In examining a large residual, it will sometimes appear that it is so evidently due to a "natural mistake" that it may be corrected without a doubt from the evidence furnished by the other observations, and the discrepant observation changed so that it may be treated as a good one. Thus, an angle may be read 5′ or 10′ wrong, or a micrometer screw may be read 5 or 10 revolutions out of the way, as shown by the rest of the observations; and the like. Such corrections should be made with great caution, however, especially if the number of observations is small. In the precise leveling of the Coast and Geodetic Survey, the observer is required to run over each section of the line twice. If a discrepancy between two results is discovered that is greater than is allowable and which is evidently due to a natural mistake, the observer is not allowed, no matter how plain

the case may be, to correct the mistake and continue. He must rerun the section until he secures two results which are within the required limit and are not subject to any assumption as to a natural mistake.

73. Again, the computer, instead of trusting to his judgment, may call in the aid of the calculus of probabilities, and seek to establish a test or criterion for the rejection of observations which will serve for all kinds of observations. Of the criterions which have been proposed the earliest is due to Professor Peirce. It is as follows: "Observations should be rejected when the probability of the system of errors obtained by retaining them is less than that of the system of errors obtained by their rejection multiplied by the probability of making so many and no more abnormal observations." A proof by Dr. Gould will be found in the *U. S. Coast Survey Report*, 1854, pp. 131, 132. It is founded on the assumption of the Gaussian law of error.

Another criterion "for the rejection of one doubtful observation" is given by Chauvenet in his *Astronomy*, vol. ii. p. 565. "We have seen that the function (Art. 30)

$$\frac{2h}{\sqrt{\pi}}\int_0^{\rho\frac{a}{r}} e^{-h^2\Delta^2}d\Delta$$

represents in general the number of errors less than a which may be expected to occur in any extended series of observations when the whole number of observations is taken as unity, r being the p. e. of an observation. If this be multiplied by the number of observations n, we shall have the actual number of errors less than a; and hence the quantity

$$n - n\,\Theta(t) = n\{1 - \Theta(t)\}$$

expresses the number of errors to be expected greater than the limit a. But if this quantity is less than $\frac{1}{2}$ it will follow that an error of the magnitude a will have a greater probability against it than for it, and may, therefore, be rejected. The

limit of rejection of *a single doubtful observation* is, therefore, obtained from the equation

$$\tfrac{1}{2} = n\{1 - \Theta(t)\}$$

or

$$\Theta(t) = \frac{2n - 1}{2n}.$$

A third criterion was proposed by Mr. Stone, Radcliffe observer at Oxford, Eng., in *Month. Not. Roy. Astron. Soc.*, 1868, 1873, in these terms: "I assume that a particular person, with definite instrumental means and under given circumstances, is likely to make, on an average, one mistake in the making and registering n observations of a given class. The probability, therefore, is that any record of his of this class of observations as a mistake is $\frac{1}{n}$. From the average discordances among the registered observations of this class we can find the p. e. of an observation in the usual way, and also the probability of an error greater than a given quantity, as C. Then if the probability in favor of a discordance as large as C is less than that of a mistake, or $\frac{1}{n}$, the discordant observation is rejected."

CHAPTER IV.

ADJUSTMENT OF INDIRECT OBSERVATIONS OF ONE UNKNOWN.

Determination of the Most Probable Values.

74. If direct measurements of a quantity have been made under the same circumstances, we have seen that the arithmetic mean of these measures gives the most probable value of the quantity. We now come to the case where the quantity measured is not the unknown required, but is a linear function of one or more unknowns whose values are to be found. This is the more general form.

Let the equations connecting a series of observed quantities $M_1, M_2, \ldots M_n$, n in number, and the independent unknowns $X, Y, \ldots, n_i$ in number ($n > n_i$), be

$$\begin{aligned} a_1X + b_1Y + \cdots - L_1 &= M_1 + v_1, \\ a_2X + b_2Y + \cdots - L_2 &= M_2 + v_2, \\ &\cdots\cdots \\ a_nX + b_nY + \cdots - L_n &= M_n + v_n, \end{aligned} \qquad (1)$$

where $a_1, b_1, \ldots L_1 \ldots L_n$ are constants given by theory for each observation, and $v_1, v_2, \ldots v_n$ are the residual errors of observation.

In practice the labor of handling these equations will be much lessened by using an artifice we have several times already employed (see Art. 26). Let $X', Y', \ldots$ be close approximations to the value of $X, Y, \ldots$ found by ordinary elimination from a sufficient number of the equations, or by some other method, as by trial, for example, and put

$$X - X' = x, \ Y - Y' = y, \cdots$$

where $x, y, \ldots$ are the corrections required to reduce the approximate values to the most probable values.

Then the observation equations reduce to

$$
\begin{aligned}
a_1x + b_1y + \cdots - l_1 &= v_1, \\
a_2x + b_2y + \cdots - l_2 &= v_2, \\
\cdots &\cdots \\
a_nx + b_ny + \cdots - l_n &= v_n,
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
-l_1 &= a_1X' + b_1Y' + \cdots - L_1 - M_1, \\
-l_2 &= a_2X' + b_2Y' + \cdots - L_2 - M_2, \\
\cdots &\cdots \\
-l_n &= a_nX' + b_nY' + \cdots - L_n - M_n,
\end{aligned}
$$

and are, therefore, known quantities.

Since the principle that the sum of the squares of the residual errors is a minimum holds whether the observed quantity is a function of one or of several unknowns (Art. 12), we can apply it to the simultaneous solution of the equations.

The residual errors must satisfy the relation

$$v_1^2 + v_2^2 + \cdots + v_n^2 = \text{a min.};$$

that is, we must make

$$
\begin{aligned}
(a_1x + b_1y + \cdots - l_1)^2 + (a_2x + b_2y + \cdots - l_2)^2 \\
+ \cdots + (a_nx + b_ny + \cdots - l_n)^2 = \text{a min.}
\end{aligned}
$$

Now, the variables $x, y, \ldots$ being independent, the differential coefficients of the expression for the minimum with respect to each of them in succession must be equal to zero.

Hence

$$
\begin{aligned}
a_1(a_1x + b_1y + \cdots - l_1) + a_2(a_2x + b_2y + \cdots - l_1) + \cdots &= 0, \\
b_1(a_1x + b_1y + \cdots - l_1) + b_2(a_2x + b_2y + \cdots - l_2) + \cdots &= 0, \\
\cdots &\cdots
\end{aligned}
\tag{3}
$$

or, collecting the coefficients of $x, y, \ldots$ in each equation,

$$
\begin{aligned}
[aa]\,x + [ab]\,y + [ac]\,z + \cdots &= [al], \\
[ab]\,x + [bb]\,y + [bc]\,z + \cdots &= [bl], \\
[ac]\,x + [bc]\,y + [cc]\,z + \cdots &= [cl], \\
\cdots &\cdots
\end{aligned}
\tag{4}
$$

These equations are called *normal equations.* The number of these equations is the same as the number of unknowns; that is, n_i. Their solution will give the most probable values of $x, y, \ldots$, and adding these values to the approximate values $X', Y' \ldots$ already known, the most probable values of X, $Y, \ldots$ will result.

It is useful to notice that equations (3) may be written

$$\begin{aligned}[av] &= 0,\\ [bv] &= 0,\\ &\cdots\end{aligned} \tag{5}$$

These relations correspond to $[v] = 0$ in the case of the arithmetic mean, and may be used as a check on the computation of the values of the residuals.

Ex. — Given the elevation of Ogden above the ocean by C. P. R. R. levels to be 4301 feet, and the elevation of Cheyenne to be 6075 feet; also the elevation of Cheyenne above Ogden by U. P. R. R. levels to be 1749 feet; find the adjusted elevations of Ogden and Cheyenne above the ocean, supposing the given results to be of equal value.

Let X, Y denote the elevations of Ogden and Cheyenne respectively. Then

$$(X - 4301)^2 + (Y - 6075)^2 + (X - Y + 1749)^2 = \text{a min.}$$

Differentiate with respect to X, Y in succession, and

$$\begin{aligned} 2X - Y &= 2552\\ -X + 2Y &= 7824.\\ \therefore X &= 4309 \text{ feet.}\\ Y &= 6067 \text{ feet.}\end{aligned}$$

75. If the observation equations are of different weights $p_1, p_2, \ldots p_n$, then, reducing each equation to the same unit of weight by multiplying it by the square root of its weight, we have (Art. 48),

$$\begin{aligned}\sqrt{p_1}a_1x + \sqrt{p_1}b_1y + \cdots - \sqrt{p_1}l_1 &= \sqrt{p_1}v_1,\\ \sqrt{p_2}a_2x + \sqrt{p_2}b_2y + \cdots - \sqrt{p_2}l_2 &= \sqrt{p_2}v_2,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\ \sqrt{p_n}a_nx + \sqrt{p_n}b_ny + \cdots - \sqrt{p_n}l_n &= \sqrt{p_n}b_n,\end{aligned} \tag{1}$$

with

$$[pv^2] = \text{a min,}$$

Substituting the values of $\sqrt{p_1}\, v_1$, $\sqrt{p_2} v_2$, . . . in the minimum equation, and differentiating with respect to x, y, . . . as independent variables, we have the normal equations

$$\begin{aligned}[paa]\,x + [pab]\,y + \cdots &= [pal],\\ [pab]\,x + [pbb]\,y + \cdots &= [pbl],\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\tag{2}$$

from which x, y, . . . may be found.

The relations for weighted equations corresponding to those of Eq. 3, Art. 74, are evidently

$$[pav] = 0,\ [pbv] = 0, \cdots \tag{3}$$

Formation of the Normal Equations.

76. Instead of forming the minimum equation and differentiating with respect to the unknowns in succession, it is more convenient to proceed according to the following plans suggested by the form of the normal equations themselves.

The first, from equations 3, Art. 74, may be stated as follows: To form the normal equation in x, multiply each observation equation by the coefficient of x in that equation, and add the results. To form the normal equation in y, multiply each observation equation by the coefficient of y in that equation, and add the results. Similarly for the remaining unknowns.

The second is suggested by the complete form of the normal equations as given in equations 4, Art. 74. According to this plan we compute the quantities $[aa]$, $[ab]$, . . . $[al]$, etc., separately, and write in their proper places in the equations.

The equality of the coefficients of the normal equations in the horizontal and vertical rows leads to a considerable shortening of the numerical work in computing these quantities. Thus with three unknowns, x, y, z, all the unlike coefficients are contained in

$$\begin{aligned}+ [aa]\,x + [ab]\,y + [ac]\,z &= [al],\\ + [bb]\,y + [bc]\,z &= [bl],\\ + [cc]\,z &= [cl].\end{aligned}$$

Instead, therefore, of computing 12 quantities, only 9 are necessary, as the remaining 3 can be at once written down. With n unknowns the saving of computation amounts to

$$1 + 2 + 3 + \cdots + (n-1) = \tfrac{1}{2}n(n-1)$$

quantities.

If the observation equations are of different weights, the formation of the normal equations may be carried out precisely in the same way as in the preceding as soon as the observation equations have been reduced to the same unit of weight.

The form of the weighted normal equations, however, shows that it is not necessary, in order to obtain the coefficients $[paa]$, $[pab]$, . . . to multiply the observation equations by the square roots of their weights, and form the auxiliary equations 1, Art. 75, since the products aa, ab, . . . multiplied by the weights of the respective equations from which they are formed and summed, give $[paa]$, $[pab]$, . . . directly. This is important because labor-saving.

77. Ex. 1.—Given the observation equations, all of equal weight,

$$\begin{aligned} x \qquad\quad &= 1 \\ x + y \qquad &= 3 \\ x - y + z &= 2 \\ -x - y + z &= 1 \end{aligned}$$

show that the normal equations are

$$\begin{aligned} 4x + y \qquad\quad &= 5 \\ x + 3y - 2z &= 0 \\ -2y + 2z &= 3 \end{aligned}$$

Ex. 2.—The expansions x_1, x_2, x_3, x_4 for 1° Fahr. of four standards of length were found by special experiment to be connected by the following relations at a temperature of 62° Fahr. (μ = one micron.).

		μ		
$+x_1$	$=$	39.915	weight	1
$+x_2$	$=$	5.932	"	16
$+x_3$	$=$	5.371	"	4
$+x_2 - 1.0937x_3$	$=$	0.006	"	8
$+4x_2 - x_4$	$=$	$-$ 1.335	"	3
$+x_1 - x_4$	$=$	$+$ 14.833	"	6

find their most probable values.

[The normal equations are

$$
\begin{array}{rrrrl}
+7x_1 & & & -\ 6x_4 & = +128.943 \\
& +72.000x_2 & -\ 8.750x_3 & -\ 12x_4 & = +\ 78.940 \\
& -\ 8.750x_2 & +13.569x_3 & & = +\ 21.432 \\
-6x_1 & -12.000x_2 & & +\ 9x_4 & = -\ 84.993 \\
\mu & \mu & \mu & \mu &
\end{array}
$$

and $x_1 = 39.913,\ x_2 = 5.932,\ x_3 = 5.405,\ x_4 = 25.075.$]

An example will now be given to illustrate the method of forming a series of observation equations:

Ex. 3. — At Washington the meridian transits of the following stars were observed to determine the correction and rate of sidereal clock Kessels No. 1324, April 12, 1870, at 11 hours clock time.

Star.	Observed Clock Time of Transit, T.	Right Ascension of Star, a.
	h. m. s.	s.
τ Leonis.	11 21 17.98	16.00
ν Leonis.	11 30 20.41	18.51
β Leonis.	11 42 28.57	26.57
o Virginis.	11 58 38.15	36.20
η Virginis.	12 13 18.37	16.37
θ Virginis.	13 3 16.36	14.39

Let x = corr. of clock at 11 hours clock time,
y = rate per hour of clock.

Now, from theoretical considerations * it is known that the equation

$$x + y\,(T - 11) = a - T$$

gives the relation between the clock correction and rate and the clock time of transit of each star observed.

Hence the observation equations are

$$
\begin{aligned}
x + 0.35y &= -\ 1.98 \\
x + 0.50y &= -\ 1.90 \\
x + 0.71y &= -\ 2.00 \\
x + 0.98y &= -\ 1.95 \\
x + 1.22y &= -\ 2.00 \\
x + 2.05y &= -\ 1.97
\end{aligned}
$$

For the remainder of the solution, see Art. 114.

Ex. 4. — In the triangulation of Lake Superior executed by the U. S. Engineers there were measured at station Sawteeth East the angles

* See Chauvenet, *Astronomy*, vol. ii. chap. v.

Farquhar-Porcupine	62° 59′ 40.33″	weight	5
Farquhar-Outer	64° 11′ 34 92″	"	7
Farquhar-Bayfield	100° 20′ 29.12″	"	4
Porcupine-Bayfield	37° 20′ 49.55″	"	7
Outer-Bayfield	36° 08′ 55.86″	"	4

required the adjusted values of the angles.

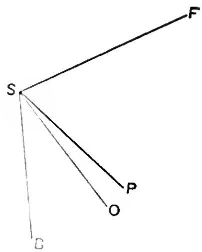

Fig. 3.

All of the angles may evidently be expressed in terms of FSP, FSO, FSB. Let X, Y, Z denote the most probable values of these angles, and let X', Y', Z' be assumed approximate values of these most probable values, and x, y, z their most probable corrections. Denoting the measured angles in order by M_1, M_2, . . . M_5, and their most probable corrections by v_1, v_2, . . . v_5, we have

$$
\begin{aligned}
X' + x &= X &&= M_1 + v_1 \\
Y' + y &= Y &&= M_2 + v_2 \\
Z' + z &= Z &&= M_3 + v_3 \\
X' - x + Z' + z &= -X + Z &&= M_4 + v_4 \\
Y' - y + Z' + z &= -Y + Z &&= M_5 + v_5
\end{aligned}
$$

For simplicity the assumed approximate values may be taken equal to the observed values of the angles, so that we have the reduced observation equations

$$
\begin{array}{rrrrll}
+x & & & = v_1 & \text{weight} & 5 \\
 & +y & & = v_2 & \text{"} & 7 \\
 & & +z & = v_3 & \text{"} & 4 \\
-x & & +z - 0.76 & = v_4 & \text{"} & 7 \\
 & -y & +z - 1.66 & = v_5 & \text{"} & 4
\end{array}
$$

Hence the normal equations

$$
\begin{array}{rrrl}
12x & & -\ 7z & = -\ \ 5.32 \\
 & +11y & -\ 4z & = -\ \ 6.64 \\
-7x & -\ 4y & +15z & = +11.96
\end{array}
$$

Solving these equations, we find

$$x = -0.05'', \quad y = -0.36'', \quad z = +0.68''.$$

Hence, $v_1 = -0.05''$, $v_2 = -0.36''$, $v_3 = +0.68''$, $v_4 = -0.03''$, $v_5 = -0.62''$,

and the adjusted values of the angles are

62° 59′ 40.28″
64° 11′ 34.56″
100° 20′ 29.80″
37° 20′ 49.52″
36° 08′ 55.24″

We might have used $v_1, v_2, \ldots v_5$, for the corrections without introducing the symbols x, y, z at all.

Ex. 5. If the unknown x occurs in each of the n observation equations

$$\begin{array}{ll} -x + b_1 y + c_1 z + \ldots = l_1 & \text{weight } 1, \\ -x + b_2 y + c_2 z + \ldots = l_2 & \quad " \quad 1, \\ \cdots\cdots\cdots\cdots & \end{array}$$

these equations are equivalent to the reduced observation equations,

$$\begin{array}{ll} b_1 y + c_1 z + \ldots = l_1 & \text{weight } 1, \\ b_2 y + c_2 z + \ldots = l_2 & \quad " \quad 1, \\ \cdots\cdots\cdots\cdots & \\ [b] y + [c] z + \ldots = [l] & \quad " \quad -\frac{1}{n}. \end{array}$$

[For the normal equations found from the first set after eliminating x are the same as the normal equations formed from the second set directly.]

Ex. 6. — Instead of the observation equation

$$ax + by + cz + \ldots = l \text{ weight } p,$$

we may write

$$qax + qby + \ldots = ql \text{ weight } \frac{p}{q^2}.$$

78. Control of the Formation of the Normal Equations. — A very convenient check or control is the following. Add as an extra term to each observation equation the sum of the coefficients of $x, y, \ldots$ and of the absolute term l in that equation, and treat these added terms just as we do the absolute terms. Thus let $s_1, s_2, \ldots s_n$ denote the sums, so that

$$\begin{array}{l} a_1 + b_1 + c_1 + \cdots + l_1 = s_1, \\ a_2 + b_2 + c_2 + \cdots + l_2 = s_2, \\ \cdots\cdots\cdots\cdots \\ a_n + b_n + c_n + \cdots + l_n = s_n. \end{array}$$

Multiply each of these expressions by its a and add the products, each by its b and add, and so on; then

$$\begin{array}{l} [aa] + [ab] + \cdots + [al] = [as], \\ [ab] + [bb] + \cdots + [bl] = [bs], \\ \cdots\cdots\cdots\cdots \\ [al] + [bl] + \cdots + [ll] = [ls]. \end{array}$$

If these equations are satisfied, the normal equations are correct. Thus each normal equation is tested as soon as it is formed.

Since $[aa]$, $[ab]$, . . . $[al]$ have been computed in forming the normal equations, the only new terms to be computed in applying the check are $[as]$, $[bs]$, . . . $[ls]$, $[ll]$.

Various modifications may readily be applied to suit individual tastes. Thus the absolute term may be placed on the other side of the sign of equality; or the sign of the check may be changed so as to make the sum of each horizontal row equal to zero.

79. Forms of Computing the Normal Equations. — When the number of unknowns in the observation equations is large, or when their coefficients contain several figures, it is convenient to have a fixed form for the computation of the terms of the normal equations. It lightens the labor much either in forming, solving, or in finding the precision of the unknowns from these equations, if the computation is so arranged that a check can at all times be applied and the whole process proceed in a uniform and mechanical manner.

The aids in the arithmetical work are a table of squares, a table of products [Crelle's], a table of reciprocals, a table of logarithms, and an arithmometer, or machine for performing multiplications and divisions. The latter is of the greatest use in computations of this kind. With it the *drudgery* of computation is in great measure got rid of. On the Lake Survey two forms of machine were used, the Grant and the Thomas. In the Coast and Geodetic Survey there are in use (1904) the Thomas or Burckhardt machine, the Brunsviga machine, and the Thatcher slide-rule.

With the Crelle multiplication tables as good speed can be made as with the machine if the number of significant figures required in the products is so small that little or no interpolation is required in the tables.

Form (*a*). With Crelle's tables, or with a machine, the products aa, ab, . . . are found directly, and all that is then to be done is to write them in columns and take their sums $[aa]$, $[ab]$, . . . With a Thomas machine, however, each product may be added to all that precede, so that the final values result at once.

Let us, for example, take the observation equations

$$\begin{aligned} -1.2\,x + 0.2\,y + 0.9 &= v_1, \\ +3.0\,x - 2.1\,y + 1.1 &= v_2, \\ +0.7\,x + 1.6\,y - 4.0 &= v_3. \end{aligned}$$

Arrange as follows, the headings indicating the nature of the numbers underneath:

a	*b*	*l*	*s*
− 1.2	+ 0.2	+ 0.9	− 0.1
+ 3.0	− 2.1	+ 1.1	+ 2.0
+ 0.7	+ 1.6	− 4.0	− 1.7

aa	*ab*	*al*	*as*
1.44	− 0.24	− 1.08	+ 0.12
9.00	− 6.30	+ 3.30	+ 6.00
0.49	+ 1.12	− 2.80	− 1.19
+ 10.93	− 5.42	− 0.58	+ 4.93
	bb	*bl*	*bs*
....	0.04	+ 0.18	− 0.02
....	4.41	− 2.31	− 4.20
....	2.56	− 6.40	− 2.72
− 5.42	+ 7.01	− 8.53	− 6.94
		ll	*ls*
....	....	0.81	− 0.09
....	....	1.21	+ 2.20
....	....	16.00	+ 6.80
− 0.58	− 8.53	+ 18.02	+ 8.91

Hence the normal equations, with the check all ready for solution, are

$$\begin{aligned} o &= +10.93\,x - 5.42\,y - 0.58 \qquad + 4.93, \\ o &= -\ \ 5.42\,x + 7.01\,y - 8.53 \qquad - 6.94. \end{aligned}$$

80. *Form (b).* If logarithms alone are used, form a table of the log coefficients of the observation equations as follows:

$$\begin{array}{l}\log a_1, \log b_1, \ldots \log l_1, \log s_1, \\ \log a_2, \log b_2, \ldots \log l_2, \log s_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \log a_n, \log b_n, \ldots \log l_n, \log s_n.\end{array}$$

Write $\log a_1$ on a slip of paper and carry it along the top row, forming the products,

$$\log a_1 a_1, \log a_1 b_1, \ldots \log a_1 l_1, \log a_1 s_1 .$$

Similarly with $\log a_2$ form the products from the second row,

$$\log a_2 a_2, \log a_2 b_2, \ldots \log a_2 l_2, \log a_2 s_2,$$

and so on till $\log a_n$ is reached.

The numbers corresponding to these logarithms are next found, so that we have

$$\begin{array}{l}a_1 a_1, a_1 b_1, \ldots a_1 l_1, a_1 s_1, \\ a_2 a_2, a_2 b_2, \ldots a_2 l_2, a_2 s_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n a_n, a_n b_n, \ldots a_n l_n, a_n s_n.\end{array}$$

By addition we find

$$[aa], [ab], \ldots [al], [as],$$

the coefficients of the unknowns in the first normal equation.

Proceed in a precisely similar way with $\log b_1$, $\log b_2$, ... b_n, omitting the term $[ab]$ already found; with $\log c_1$, $\log c_2$, ... $\log c_n$, omitting the terms $[ac]$, $[bc]$ already found; and so on till the last quantity is reached.

81. *Form* (*c*). If we wish to use a table of squares altogether, then, since

$$ab = \tfrac{1}{2}\{(a+b)^2 - a^2 - b^2\}$$

and therefore

$$[ab] = \tfrac{1}{2}\{[(a+b)^2] - [aa] - [bb]\} \tag{1}$$

we form the square sums

$$\begin{array}{r}[aa],\ [(a+b)^2],\ [(a+c)^2],\cdots[(a+l)^2],\ [(a+s)^2],\\ [bb],\ [(b+c)^2],\cdots[(b+l)^2],\ [(b+s)^2],\\ \cdots\cdots\cdots\cdots\cdots\\ [ll],\ [(l+s)^2],\end{array}$$

and perform the necessary subtractions.

In doing this, first take from the table of squares the squares aa, bb, . . . ll, ss, and sum them; next write the coefficients a of x on a slip of paper and carry them over the coefficients of y, z, . . ., forming the sums $a_1 + b_1$, $a_1 + c_1$, . . .; $a_2 + b_2$, $a_2 + c_2$, . . . Take out the squares of these numbers and sum them. Proceed similarly with the coefficients of y, z, . . . Finish as indicated in (1).

Thus in the preceding example,

aa	bb	ll	ss
1.44	0.04	0.81	0.01
9.00	4.41	1.21	4.00
0.49	2.56	16.00	2.89
10.93	7.01	18.02	6.90

$a+b$	$(a+b)^2$	$a+l$	$(a+l)^2$	$a+s$	$(a+s)^2$
1.0	1.00	0.3	0.09	1.3	1.69
0.9	0.81	4.1	16.81	5.0	25.00
2.3	5.29	3.3	10.89	1.0	1.00
	7.10		27.79		27.69
$[aa]+[bb] =$	17.94	$[aa]+[ll] =$	28.95	$[aa]+[ss] =$	17.83
	− 10.84		− 1.16		9.86
	− 5.42		− 0.58		4.93
	$= [ab]$		$= [al]$		$= [as]$

giving the same results as before.

This form, which is very neat analytically, was first given by Bessel in the *Astron. Nachr.*, No. 399.

A consideration of the simple case of three observation equations, each involving two unknowns, will show that to form the normal equations, using a log table only, 24 entries in the table

are required, while by this method we only need to enter a table of squares 18 times, thus effecting a saving of 6 entries. The Bessel method has also the advantage that, as we deal with squares, all thought with regard to sign is done away with. Besides, if the table of squares is a very extended one, accuracy can be had to a greater number of decimal places than with an ordinary log. table. As compared with the logarithmic form, then, this method is to be preferred, more especially when the coefficients are not very different.

On the other hand, if Crelle's tables or a computing machine is to be had, the direct process explained in (a) is much to be preferred to either, as experience will show.

82. It is worth noticing that whichever method of formation of the normal equations is adopted, labor will be saved by changing the units in which the unknowns are expressed if the coefficients of the different unknowns are very different. Thus, suppose we had the observation equations,

Equation	Check Sums.
$1000\,x + 0.0001\,y = 4.11$	1004.1101
$999\,x + 0.0002\,y = 3.93$	1002.9302
. . . .	

from which to find x and y.

By placing

$$x' = 100\,x,\quad y' = 0.01\,y$$

the equations reduce to

Equation	Check Sums.
$10\,x' \quad + 0.01\,y' = 4.11$	14.12
$9.99\,x' + 0.02\,y' = 3.93$	13.94
. . . .	

which are in more manageable shape for solution.

83. Before beginning the solution of a series of normal equations we should consider whether the object is to find:

(1) the unknowns only, or

(2) the unknowns and their weights;

and, in the latter case,

(a) whether the number of unknowns is large,

(b) whether many of the coefficients of the unknowns in the normal equations are wanting.

Normal equations may be solved by the ordinary algebraic methods for the elimination of linear equations or by the method of determinants. When, however, they are numerous, the method of substitution introduced by Gauss and the Doolittle method are more suitable. Each has its advantages. Both are quite mechanical in operation and are well suited for use with an arithmometer, which is as great a help in solving as it is in forming the normal equations.

84. The Method of Substitution. — For convenience in writing, take three unknowns, x, y, z, the process being the same whatever the number.

The normal equations are

$$\begin{aligned} [aa]x + [ab]y + [ac]z &= [al], \\ [ab]x + [bb]y + [bc]z &= [bl], \\ [ac]x + [bc]y + [cc]z &= [cl]. \end{aligned} \tag{1}$$

From the first equation

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z + \frac{[al]}{[aa]}. \tag{2}$$

Substitute this value in the remaining equations, and, in the convenient notation of Gauss, there result

$$\begin{aligned} [bb.1]y + [bc.1]z &= [bl.1], \\ [bc.1]y + [cc.1]z &= [cl.1], \end{aligned} \tag{3}$$

where

$$\begin{aligned} [bb.1] &= [bb] - \frac{[ab]}{[aa]}[ab], \\ [bc.1] &= [bc] - \frac{[ab]}{[aa]}[ac], \\ [bl.1] &= [bl] - \frac{[ab]}{[aa]}[al], \\ [cc.1] &= [cc] - \frac{[ac]}{[aa]}[ac], \end{aligned} \tag{4}$$

$$[cl.1] = [cl] - \frac{[ac]}{[aa]}[al].$$

Again, from the first of equations 3,

$$y = -\frac{[bc.1]}{[bb.1]}z + \frac{[bl.1]}{[bb.1]}, \tag{5}$$

which value substituted in the second equation gives

$$z = \frac{[cl.2]}{[cc.2]}, \tag{6}$$

where

$$\left.\begin{aligned}[cc.2] &= [cc.1] - \frac{[bc.1]}{[bb.1]}[bc.1], \\ [cl.2] &= [cl.1] - \frac{[bc.1]}{[bb.1]}[bl.1].\end{aligned}\right\} \tag{7}$$

Having thus found z, we have y at once by substituting in (5), and thence x by substituting for y and z their values in (2).

The first equations of the successive groups in the elimination collected are

$$\begin{aligned}[aa]x + [ab]y + [ac]z &= [al], \\ [bb.1]y + [bc.1]z &= [bl.1], \\ [cc.2]z &= [cl.2].\end{aligned} \tag{8}$$

These are called the *derived normal equations*.

Divide each of these equations by the coefficient of its first unknown, and

$$\begin{aligned}x + \frac{[ab]}{[aa]}y + \frac{[ac]}{[aa]}z &= \frac{[al]}{[aa]}, \\ y + \frac{[bc.1]}{[bb.1]}z &= \frac{[bl.1]}{[bb.1]}, \\ z &= \frac{[cl.2]}{[cc.2]}.\end{aligned} \tag{9}$$

85. Controls of the Solution. — In solving a set of normal equations a control is essential. It is sometimes recommended to solve the equations arranged in the reverse order, when, if the

work is correct, the same results will be found as before. But what is wanted in a control is a means of checking the work at each step, and not at the conclusion, it may be, of several weeks' work, when, if the results do not agree, all that is known is that there is a mistake somewhere without being able to locate it.

(a) Continuation of the formation control. Experience has shown that it is convenient to carry on through the solution the check used in forming the equations. It merely consists in placing as an extra term to each equation the sums $[as]$, $[bs]$, . . . $[ls]$, and operating on them in the same way as on the absolute terms $[al]$, $[bl]$, . . . The sum of the terms in every line, after each elimination of an unknown, must be each equal to the check sum numerically; the closeness of the agreement depending on the number of decimal places employed.

This check may be applied at every step and mistakes be weeded out.

(b) The diagonal coefficients $[aa]$, $[bb]$, . . . of the normal equations, and $[aa]$, $[bb.1]$, $[cc.2]$, . . . of the derived normal equations, are always positive.

For $[aa]$, $[bb.1]$, . . . being the sums of squares, are positive. Also

$$[aa], [bb.1] = \begin{vmatrix} [aa], & [ab] \\ [ab], & [bb] \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1, & b_1 \\ a_3, & b^3 \end{vmatrix}^2 + \cdot \cdot \cdot,$$

a positive quantity.

(c) By equations 5, Art. 74, the residuals found by substituting for x, y, z their values in the observation equations must satisfy the relations

$$[av] = [bv] = \cdot \cdot \cdot = 0.$$

(d) A very complete check is afforded by the different methods of computing $[vv]$ the sum of the squares of the residuals. (See Art. 106.)

86. Forms of Solution. — In applying the method of substitution to any special example it is important that the arrange-

ment of the computation be convenient and that every step be written down. Experience teaches that simplicity and uniformity of operation are great safeguards against mistakes.

Form (a). Solution without logarithms.

The following form has been found by experience to be convenient. It is well fitted for use with the arithmometer or any other rapid method of multiplication. The form can be readily modified to suit computer's tastes.

For illustration let us take, as before, three unknowns, x, y, z. The computation is divided into sections, each section being formed in a precisely similar way, and in each section one unknown is eliminated.

Given the normal equations,

No.	x	y	z		CHECK.	REMARKS.
I.	$[aa]$	$[ab]$	$[ac]$	$[al]$	$[as]$	. . . . .
II.	$[ab]$	$[bb]$	$[bc]$	$[bl]$	$[bs]$	. . . . .
III.	$[ac]$	$[bc]$	$[cc]$	$[cl]$	$[cs]$	. . . . .
			Solution.			
IV.	1	$\frac{[ab]}{[aa]}$	$\frac{[ac]}{[aa]}$	$\frac{[al]}{[aa]}$	$\frac{[as]}{[aa]}$	I. ÷ $[aa]$
V.		$[ab]\frac{[ab]}{[aa]}$	$[ab]\frac{[ac]}{[aa]}$	$[ab]\frac{[al]}{[aa]}$	$[ab]\frac{[as]}{[aa]}$	IV. × $[ab]$
II.		$[bb]$	$[bc]$	$[bl]$	$[bs]$	II.
VI.		$[bb.1]$	$[bc.1]$	$[bl.1]$	$[bs.1]$	II. — V.
VII.		$[ac]\frac{[ab]}{[aa]}$	$[ac]\frac{[ac]}{[aa]}$	$[ac]\frac{[al]}{[aa]}$	$[ac]\frac{[as]}{[aa]}$	IV. × $[ac]$
III.		$[bc]$	$[cc]$	$[cl]$	$[cs]$	III.
VIII.		$[bc.1]$	$[cc.1]$	$[cl.1]$	$[cs.1]$	III. — VII.
IX.		1	$\frac{[bc.1]}{[bb.1]}$	$\frac{[bl.1]}{[bb.1]}$	$\frac{[bs.1]}{[bb.1]}$	VI. ÷ $[bb.1]$
X.			$[bc.1]\frac{[bc.1]}{[bb.1]}$	$[bc.1]\frac{[bl.1]}{[bb.1]}$	$[bc.1]\frac{[bs.1]}{[bb.1]}$	IX. × $[bc.1]$
VIII.			$[cc.1]$	$[cl.1]$	$[cs.1]$	
XI.			$[cc.2]$	$[cl.2]$	$[cs.2]$	VIII. — X.
			1	$\frac{[cl.2]}{[cc.2]}$	$\frac{[cs.2]}{[cc.2]}$	XI. ÷ $[cc.2]$

$$\therefore z = \frac{[cl.2]}{[cc.2]}$$

From Eq. IX.
$$y = -z\frac{[bc.1]}{[bb.1]} + \frac{[bl.1]}{[bb.1]}.$$

From Eq. IV.
$$x = -y\frac{[ab]}{[aa]} - z\frac{[ac]}{[aa]} + \frac{[ab]}{[aa]}.$$

To eliminate the first unknown, x. In the first line write the quotients $\frac{[ab]}{[aa]}$, $\frac{[ac]}{[aa]}$, . . . that is, the coefficients of the first normal equation divided by $[aa]$, the coefficient of x in that equation.

The first line is now multiplied in order by $[ab]$, $[ac]$, forming the second and fifth lines.

In the third and sixth lines write equations II. and III.

The fourth line is the sum of the second and third, and the seventh the sum of the fifth and sixth.

This concludes the elimination of x, and the results in the fourth and seventh lines involve y and z only.

Take now these results and proceed in a precisely similar way to eliminate y.

The value of the last unknown, z, next results.

Now proceed to find y and x. Thus, substitute for z its value in the eighth line, and we have y; and for y and z their values in the first line, and we have x.

87. In carrying this solution into practice, there are three points that deserve special notice:

(1) In order to render the work mechanical, and so lighten the labor, the number of different operations should be made as small as possible. Instead, therefore, of dividing by $[aa]$, $[bb.1]$, $[cc.2]$, it is better to multiply by the reciprocals of these quantities, and, in order to avoid subtractions, to first change the signs of the reciprocals. We shall then have to perform only two simple operations — multiplication and addition. By transferring the terms $[al]$, $[bl]$, $[cl]$ to the left-hand side of the equations before beginning the solution, the values of the unknowns will come out with their proper signs.

(2) Equations VI. and VIII. are the normal equations with x eliminated. An inspection of them shows that the coefficients of the unknowns follow the same law as the coefficients of the unknowns in the original normal equations with respect to symmetry of vertical and horizontal columns. Hence in the elimination it is unnecessary to compute these common terms more than once. Thus $[bc.1]$ from Eq. VI. may be written down as the first term of Eq. VIII. This principle is of great use in shortening the work when the number of unknowns is large.

(3) In a numerical example it is evident that since $[aa]$, $[bb.1]$, $[cc.2]$ do not in general divide exactly into the other coefficients of their respective equations, and that only approximate values of the unknowns can at best be obtained, it will give a closer result to divide by the larger coefficients and multiply by the smaller than *vice versa*. Attention to this by a proper arrangement of the coefficients before beginning the solution results in a considerable saving of labor, as the successive coefficients in the course of the elimination need not be carried to as many places of decimals to insure the same accuracy that a different arrangement would require.

Ex.—To make the preceding perfectly plain we shall solve in full the normal equations formed in Art. 79.

(1) Write the absolute term on the right of the sign of equality, and make the check sum equal to the sum of the other terms in each horizontal row.

	x	y	l	CHECK.	REMARKS.
I.	+ 10.93	− 5.42	+ 0.58	+ 6.09	. . . . .
II.	− 5.42	+ 7.01	+ 8.53	+ 10.12	. . . . .
III.	+ 1	− 0.496	+ 0.053	+ 0.557	I. ÷ 10.93
IV.	. . . .	+ 2.688	− 0.288	− 3.019	III. × − 5.42
II.	. . . .	+ 7.01	+ 8.53	+ 10.120	II.
V.	. . . .	+ 4.322	+ 8.818	+ 13.139	II. − IV.
VI.	. . . .	+ 1	+ 2.040 = y	+ 3.040	V. ÷ 4.322
VII.	. . . .	− 0.496	− 1.012	− 1.508	VI. × − 0.496
III.	+ 1	− 0.496	+ 0.053	+ 0.557	III.
VIII.	+ 1	0	+ 1.065 = x	+ 2.065	III. − VII.

Hence $x = 1.06$ $y = 2.04$

(2) Write the constant term on the left of the sign of equality, and form the check so as to make the sum of the terms in each horizontal line equal to zero.

	Reciprocals.	x	y	l	Check.	Remarks.
I.	0.0915	+ 10.93	− 5.42	− 0.58	− 4.93	. . . .
II.	. . . .	− 5.42	+ 7.01	− 8.53	+ 6.94	. . . .
III.	. . . .	− 1	+ 0.496	+ 0.053	+ 0.451	I. × − 0.0915
IV.	. . . .	. . .	− 2.688	− 0.288	− 2.445	III. × − 5.42
II.	. . . .	. . .	+ 7.010	− 8.530	+ 6.940	II.
V.	0.2314	. . .	+ 4.322	− 8.818	+ 4.495	II. + IV.
VI.	. . . .	. . .	− 1	+ 2.040 = y	− 1.040	V. × − 0.2314
VII.	. . . .	. . .	− 0.496	+ 1.012	− 0.516	VI. × 0.496
III.	. . . .	− 1	+ 0.496	+ 0.053	+ 0.451	III.
VIII.	. . . .	− 1	. . .	+ 1.065 = x	− 0.065	III. + VII.

In order to find the values of the unknowns to two places of decimals, the computation should be carried through to three places, and the third place dropped in the final result.

88. *Form (b).* The logarithmic solution.

As an example of the logarithmic method let us take the general form of the preceding example, when R and S are substituted for the absolute terms 0.58 and 8.53 respectively.

In the numerical work it is better to convert all the divisions into multiplications. Therefore write down the complementary logs. of the divisors with the signs changed. Each multiplier may now be written as needed on a slip of paper and carried over each logarithm to be operated on. Thus for the first operation the slip would have on it 8.96138 n, where the n indicates that the number is negative.

Paper ruled into small squares, so as to bring the figures in the same vertical columns and facilitate additions and subtractions, renders the work more mechanical, and is consequently an assistance to the computer.

In general solutions, when the number of unknowns is large, it will be found much better to carry a double check, one for the coefficients of x, y, . . . and the other for the coefficients of R,

S, . . . Though unnecessary in our example, it is inserted for illustration.

It will be noticed that the coefficients of R, S in the values of x, y follow the same law of symmetry as the normal equations. A little consideration will show that this is always the case.

Hence, attending to this, we may shorten the computation by leaving out the common terms. We have, therefore, one term less to compute for each unknown, proceeding from the last to the first. The case is precisely analogous to that of Art. 76.

	x	y	CHECK.	R	S	CHECK.	REMARKS.
I.	10.93	− 5.42	− 5.51	− 1	. . .	+ 1	. . . . .
II.	− 5.42	+ 7.01	− 1.59	. . .	− 1	+ 1	. . . . .
III.	1.03862	0.73400 n	0.74115 n	0	. . .	0	log I.
IV.	(8.96138 n)	9.69538	9.70253	8.96138	. . .	8.96138 n	III. − log 10.93
				+ 0.091		− 0.091	Nos.
V.	. . .	0.42938 n	0.43653 n	9.69538 n	. . .	9.69538	IV. − log 5.42
VI.	. . .	− 2.688	− 2.732	− 0.496	. . .	+ 0.496	Nos.
II.	. . .	+ 7.010	− 1.590	. . .	− 1	+ 1	II.
VII.	. . .	+ 4.322	− 4.322	− 0.496	− 1	+ 1.496	VI. + II.
VIII.	. . .	0.63568	0.63568 n	0.69548 n	0.0	0.17493	log. VII.
IX.	. . .	(9.36432 n)	0.00000	9.05980	9.36432	9.53925 n	VIII. − log 4.322
X.	. . .	− 1	+ 1	+ 0.115	+ 0.231	− 0.346	Nos.
XI.	. . .	. . .	. . .	8.75518	9.05970	9.23463 n	IX. + log $\frac{5.42}{10.93}$
XII.	. . .	. . .	. . .	+ 0.057	+ 0.115	− 0.172	Nos.
	. . .	. . .	. . .	+ 0.091	. . .	− 0.091	. . . . .
XIII.	− 1	. . .	+ 1	+ 0.148	+ 0.115	− 0.263	. . . . .

$$\therefore x = 0.148\, R + 0.115\, S,$$
$$y = 0.115\, R + 0.231\, S.$$

Substituting for R and S their values, we have, as before,

$$x = 1.06,$$
$$y = 2.04.$$

89. Ex. 1.—In the elimination of n normal equations by the method of substitution, show that the total number of independent coefficients in the original and derived normal equations is $\frac{n(n+1)(n+5)}{6}$.

[The sum is $\frac{1}{2}\{1.4 + 2.5 + \ldots n(n+3)\}$.]

Ex. 2.—If the elimination of the unknowns in the normal equations is carried out by the method of substitution, the product

$$[aa],\ [bb.1],\ [cc.2]\ \ldots$$

has the same value whatever order has been followed.

90. A method of indirect elimination by successive trials and approximations has been suggested by Gauss. It will be found in *Coast Survey Report*, 1855, Appendix 44; von Freeden, *Die Praxis der Methode der kleinsten Quadrate*, p. 96; Vogler, *Ausgleichungsrechnung*, p. 129.

It is not given, because our experience has been that it is in general intolerably tedious.

91. The Doolittle Method of Solution. — This method of solution is due to Mr. M. H. Doolittle of the Computing Division of the Coast and Geodetic Survey.* In it there is a combination of improvements on the Gaussian method of substitution. Its advantage lies mainly in the arrangement of the work in the most convenient form for the computer. This makes the solution more rapid than by the other method, the gain in speed being the more marked the greater the number of equations.

In order to make the process employed readily followed, the solution of the three normal equations,

$$\begin{aligned}[aa]x + [ab]y + [ac]z &= [al],\\ [ab]x + [bb]y + [bc]z &= [bl],\\ [ac]x + [bc]y + [cc]z &= [cl],\end{aligned}$$

is given in general terms according to this form.

The coefficients and absolute term of the first equation are written in line 1, Table A. The reciprocal of the diagonal coefficient $[aa]$ is taken from a table of reciprocals and entered in the front column with the minus sign prefixed. The remaining terms of line 1 are multiplied by this reciprocal, and the products written in line 2. This gives x as an explicit function of y and z.

The coefficients and absolute term of Eq. 2 (omitting the coefficient of x) are written in line 1, Table B. The terms in line 2, Table A, beginning with that under y, are multiplied by $[ab]$, the coefficient of y, and the products set down in line 2, Table B. The sum of lines 1, 2, Table B, is now written in line 3, Table A.

* See Appendix 8, C. and G. Survey Report for 1878, pp. 115–118.

Line 4, Table A, is found from line 3 in the same way as line 2 was found from line 1. This gives y as an explicit function of z.

The coefficients and absolute term of Eq. 3 (omitting the coefficients of x and y) are written in line 3, Table B. The terms in lines 2, 4, Table A, beginning with those under z, are multiplied by $[ac]$, $[bc.1]$, the coefficients of z in lines 1, 3 respectively, and the products set down in lines 4, 5, Table B. The sum of lines 3, 4, 5, Table B, is written in line 5, Table A.

Line 6, Table A, gives the value of z.

The next step is to find y and x. The coefficients of the explicit functions are written in Table C. The absolute terms of the explicit functions are written in the first line of Table D. The value of z is multiplied by the coefficients of z in Table C, and the products written in the second line of Table D. The sum of the numbers in column y gives the value of y written underneath in line 3. The value of y is multiplied by the coefficients of y in Table C, and the products written in the third line of Table D. The sum of the numbers in column x gives the value of x.

92. The values of x, y, z are now found to three places of decimals. Denote them by x', y', z'. If these values are not sufficiently close, a second approximation must be made. This we proceed to describe.

First substitute the values obtained in the original normal equations, and carry out to a sufficient number of decimal places. The residuals are written in the first line of Table E. The coefficients in line 1, Table C, are multiplied by $-[al]_1$, and the products written in line 2, Table E. The first reciprocal in Table A is multiplied by the same residual, and the product written in column x, line 1, Table F. The sum of the numbers in column 2, Table E, is written underneath, as $-[bl.1]_1$.

The coefficient in line 2, Table C, is multiplied by $-[bl.1]_1$ and the product written in line 3, Table E. The second reciprocal in Table A is multiplied by the same residual, and the product written in column y, line 1, Table F. The sum of

the numbers in column 3, Table E, is written underneath, as $-[cl.2]_1$.

The third reciprocal of Table A is multiplied by this residual, and the product written in column z, line 1, Table F. This gives the correction to the value of z. The first line of Table F corresponds to the first line of Table D, and exactly the same process employed in Table D will complete Table F. The total values now are

$$x = x' + x'',$$
$$y = y' + y'',$$
$$z = z' + z''.$$

A

Recip.	x	y	z	
	$+[aa]$	$+[ab]$	$+[ac]$	$-[al]$
$-\frac{1}{[aa]}$	$x=$	$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$	$+\frac{[al]}{[aa]}$
	. . .	$[bb.1]$	$+[bc.1]$	$-[bl.1]$
$-\frac{1}{[bb.1]}$	. . .	$y=$	$-\frac{[bc.1]}{[bb.1]}$	$+\frac{[bl.1]}{[bb.1]}$
	. . .	. . .	$+[cc.2]$	$-[cl.2]$
$-\frac{1}{[cc.2]}$	. . .	. . .	$z=$	$+\frac{[cl.2]}{[cc.2]}$

B

y	z	
$+[bb]$	$+[bc]$	$-[bl]$
$-\frac{[ab]}{[aa]}[ab]$	$-\frac{[ac]}{[aa]}[ab]$	$+\frac{[al]}{[aa]}[ab]$
. . .	$[cc]$	$-[cl]$
. . .	$-\frac{[ac]}{[aa]}[ac]$	$+\frac{[al]}{[aa]}[ac]$
. . .	$-\frac{[bc.1]}{[bb.1]}[bc.1]$	$+\frac{[bl.1]}{[bb.1]}[bc.1]$

C

y	z
$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$
. . .	$-\frac{[bc.1]}{[bb.1]}$

D

x	y	z
$+\frac{[al]}{[aa]}$	$+\frac{[bl.1]}{[bb.1]}$	$+\frac{[cl.2]}{[cc.2]}$
$-\frac{[ac]}{[aa]}z'$	$-\frac{[bc.1]}{[bb.1]}z'$	z'
$-\frac{[ab]}{[aa]}y'$	y'	
x'		

E

x	y	z
$-[al]_1$	$-[bl]_1$	$-[cl]_1$
. . .	$+\frac{[ab]}{[aa]}[al]_1$	$+\frac{[ac]}{[aa]}[al]_1$
. . .	$-[bl.1]_1$	$+\frac{[bc.1]}{[bb.1]}[bl.1']$
. . .	. . . .	$-[cl.2]_1$

F

x	y	z
$+\frac{[al]_1}{[aa]}$	$+\frac{[bl.1]_1}{[bb.1]}$	$+\frac{[cl.2]_1}{[cc.2]}$
$-\frac{[ac]}{[aa]}z''$	$-\frac{[bc.1]}{[bb.1]}z''$	z''
$-\frac{[ab]}{[aa]}y''$	y''	. . . .
x''	. . . .	. . . .

93. Addition of New Equations. — It often happens that after the adjustment of a long series of observations, additional observations are made leading to additional equations. To make a solution *de novo* is necessary, but the work may be very materially shortened by the process just given. Suppose, for simplicity, that one new condition has been established. This will give one additional normal equation which may be written

$$[ad]x + [bd]y + [cd]z + [dd]w = [dl], \tag{1}$$

w being the new unknown.

The extra term to each of the other normal equations may be written down at sight. The complete equations are

$$\begin{aligned}
[aa]x + [ab]y + [ac]z + [ad]w &= [al],\\
+ [bb]y + [bc]z + [bd]w &= [bl],\\
+ [cc]z + [cd]w &= [cl],\\
+ [dd]w &= [dl].
\end{aligned}$$

Now, values of x, y, z have been already found from the normal equations resulting from the original condition equations, and these values may be taken as first approximations to the values of x, y, z resulting from the above four normal equations. Substitute in (1), and

$$[ad]x' + [bd]y' + [cd]z' + [dd]w = [dl]', \tag{2}$$

where x', y', z' are corrections to the approximate values of x, y, z. The solution is now finished as follows:

Form Table C (a) by adding the extra column w to Table C.

The term $-\frac{[ad]}{[aa]}$ is found by multiplying $[ad]$ by the first reciprocal. The coefficients of the new equation, (2), are written in the first line of Table G. Since corrections to values already found are required, the method of proceeding must be similar to that employed in Table E. The notation in Tables C (a) and G explains this.

The reciprocal of the sum of column w, that is, $[dd.3]$, in Table G is written last in the column of reciprocals of Table C (a) with the minus sign. The product of this reciprocal and the absolute term $-[dl]'$ of the new equation, that is, $\frac{[dl]'}{[dd.3]}$, is an approximate value of w. This value of w is multiplied by the terms in the last column of Table C (a), and the products are written in the first line of Table H. Column z gives the correction to z. Table H is now completed in the same way as Tables D and F.

C (a)

	y	z	w
$-\frac{1}{[aa]}$	$-\frac{[ab]}{[aa]}$	$-\frac{[ac]}{[aa]}$	$-\frac{[ad]}{[aa]}$
$-\frac{1}{[bb.1]}$	...	$-\frac{[bc.1]}{[bb.1]}$	$-\frac{[bd.1]}{[bb.1]}$
$-\frac{1}{[cc.2]}$	...	...	$-\frac{[cd.2]}{[cc.2]}$
$-\frac{1}{[dd.3]}$	...	...	...

G

x	y	z	w
$[ad]$	$[bd]$	$[cd]$	$[dd]$
...	$-\frac{[ab]}{[aa]}[ad]$ $[bd.1]$	$-\frac{[ac]}{[aa]}[ad]$	$-\frac{[ad]}{[aa]}[ad]$
...	....	$-\frac{[bc.1]}{[bb.1]}[bd.1]$ $[cd.2]$	$-\frac{[bd.1]}{[bb.1]}[bd.1]$
...	....	.....	$-\frac{[cd.2]}{[cc.2]}[cd.2]$ $[dd.3]$

H

x	y	z	w
$-\frac{[ad]}{[aa]}w$	$-\frac{[bd.1]}{[bb.1]}w$	$-\frac{[cd.2]}{[cc.2]}w$	$+\frac{[dl]'}{[dd.3]}$
$-\frac{[ac]}{[aa]}z'$	$-\frac{[bc.1]}{[bb.1]}z'$	z'	w
$-\frac{[ab]}{[aa]}y'$	y'	....	....
x'	....	....	....

94. In order to illustrate this method still further, the solution of the following equations is shown in the form indicated on the preceding pages.

Suppose given the normal equations:

$$
\begin{aligned}
1.\quad 0 &= +5.4237\,w + 2.1842\,x - 4.3856\,y + 2.3542\,z - 3.6584,\\
2.\quad 0 &= +2.1842\,w + 6.9241\,x - 1.2130\,z + 2.8563,\\
3.\quad 0 &= -4.3856\,w + 12.8242\,y + 3.4695\,z + 8.7421,\\
4.\quad 0 &= +2.3542\,w - 1.2130\,x + 3.4695\,y + 7.1243\,z + 0.6847.
\end{aligned}
$$

The solution is conducted as follows:

A

1	2	3	4	5	6	7	8
			w	x	y	z	
1	1	. . .	+5.424	+ 2.184	− 4.386	+ 2.354	−3.658
2	2	−.184	$w =$	− .401	+ .807	− .433	+ .6731
3	5	. . .	. . .	+6.048	+1.759	− 2.157	+4.323
4	6	−.165	. . .	$x =$	− .29	+ .356	− .7133
. .	. .	. . .	. . .	. . . .	. . . .	. . . .	. . . .
5	10	. . .	. . .	. . . .	+8.77	+ 5.996	+4.538
6	11	−.114	. . .	. . . .	$y =$	− .684	− .5173
. .	. .	. . .	. . .	. . . .	. . . .	. . . .	. . . .
7	16	. . .	. . .	. . . .	. . . .	+1.236	+0.704
8	17	−.809	. . .	. . . .	. . . .	$z =$	− .57

B

1	2	3	4	5	6
		x	y	z	
1	3	+6.924	. . . .	− 1.213	+ 2.856
2	4	− .876	+ 1.759	− .944	+ 1.467
3	7	. . .	+12.82	+ 3.470	+ 8.742
4	8	. . .	− 3.54	+ 1.900	− 2.950
5	9	. . .	− .51	+ .626	− 1.254
. .	. .	. . .	. . . .	. . . .	. . . .
6	12	. . .	. . . .	+7.124	+ 0.685
7	13	. . .	. . . .	− 1.019	+ 1.584
8	14	. . .	. . . .	− .708	+ 1.539
9	15	. . .	. . . .	− 4.101	− 3.104

The first column in each of the above tables gives the number of the line, and the second the order of procedure.

It is to be observed that the numbers in Table B have but a single use, while those of Table A are used over and over; and when the number of equations is large, it is of great advantage that they should be thus tabulated by themselves in a form compact and easy of reference.

C

	Reciprocal.	x	y	z
1	− .184	− .401	+ .807	− .433
2	− .165	. . .	− .29	+ .356
3	− .114	. . .	. . .	− .684
4	− .809	. . .	. . .	. . .
. . .	. . .	. . .	. . .	. . .

D

	w	x	y	z
1	+ .6731	− .7133	− .5173	− .57
2	+ .2468	− .2029	+ .3809	$= z_1$
3	.1024	+ .0368	− .127	$= y_1$
4	+ .3524	− .879	. . .	$= x_1$
. . .	+ 1.17	. . .	. . .	$= w_1$

	E					F			
	1	2	3	4		w	x	y	z
1	— .0165	+ .0169	+ .0047	+ .0039	1	+ .0030	— .0039	+ .0018	— .0239
2	. . .	+ .0066	— .0133	+ .0071	2	+ .0103	— .0085	+ .0163	$= z_2$
3	. . .	+ .0235	— .0068	+ .0084	3	+ .0146	— .0052	+ .0181	$= y_2$
4	. . .	. . .	— .0154	+ .0102	4	+ .0071	— .0176	. . .	$= x_2$
. . .	. . .	. . .	. . .	+ .0296	. . .	+ .0350	. . .	. . .	$= w_2$

As the multiplication is performed by Crelle's Tables, no multiplier is allowed to extend beyond three significant figures. Other numbers may be extended to four; but it would be a waste of time to extend any number farther, except in the process of substitution for the determination of residuals.

If an arithmometer is used, four significant figures may be retained throughout. It will seldom be advisable to retain more significant figures than this.

For the sake of perspicuity in explanation and convenience in printing, we have here made some slight departures from actual practice. For instance, in the solution of a large number of equations, it would be inconvenient to pass the eye and hand out to a vertical column of reciprocals; and they are better written in an oblique line near the quantities from which they are derived and with which they are to be employed.

By this process, Mr. Doolittle solved in five and one-half days, or 36 working hours, with far greater than requisite accuracy, 41 equations containing 174 side coefficients counting each but once, or 430 terms in all.

95. Suppose that after the solution of the foregoing equations and the consequent adjustment a new condition is established, resulting in the following normal equation: $0 = -2.0475\,w + 0.8362\,x + 1.8567\,y - 1.3149\,z + 8.2527\,u - 1.8372$; with the addition of the term $-2.0475\,u$ to the first of the previous equations, $+0.8362\,u$ to the second, etc. The absolute term 1.8372 is supposed not to be an original discrepancy, but an out-

standing residual, after the foregoing solution has fully entered into the adjustment, as is generally the case with azimuth and length equations, in a triangulation adjustment.

C (a)					
	Reciprocal.	x	y	z	u
1	− .184	− .401	+ .807	− .433	+ .377
2	− .165	. . .	− .29	+ .356	− .274
3	− .114	. . .	. . .	− .684	+ .0307
4	− .809	. . .	. . .	. . .	− .282
5	− .145	. . .	. . .	. . .	. . .
. .	. . .	. . .	. . .	. . .	. . .

G					
	w	x	y	z	u
1	− 2.05	+ .836	+ 1.851	− 1.315	+ 8.253
2	. . .	+ .822	− 1.654	+ .888	− .773
3	. . .	+ .166	− .481	+ .591	− .455
4	. . .	. . .	− .269	+ .184	− .008
5	. . .	. . .	. . . .	+ .348	− .098
. .	. . .	. . .	. . . .	. . . .	+ 6.919

H					
	w	x	y	z	u
1	+ .1007	− .0732	+ .0082	− .0753	+ .267
2	+ .0326	− .0268	+ .0515	$= z_3$	$= u_1$
3	+ .0482	− .0173	+ .0597	. . . . .	$= y_3$
4	+ .0469	− .117	. . . . .	. . . . .	$= x_3$
5	+ .228	. . . . .	. . . . .	. . . . .	$= w_3$

The Precision of the Most Probable (Adjusted) Values.

96. The problem now before us is to find the p. e. of the unknowns, $x, y, \ldots$ as determined from a series of normal equations. If the observation equations are reduced to the same unit of weight, which we shall take to be unity for convenience, the general form of the normal equations is

$$\begin{aligned} [aa]x + [ab]y + \cdots &= [al], \\ [ab]x + [bb]y + \cdots &= [bl]. \\ \cdots\cdots\cdots\cdots & \end{aligned} \qquad (1)$$

Let $\qquad r =$ the p. e. of a single observation.

$r_x, r_y, \ldots =$ the p. e. of $x, y, \ldots$

$p_x, p_y, \ldots =$ the weights of $x, y, \ldots$

From Art. 47 we have

$$p_x r_x^2 = p_y r_y^2 = \cdot \cdot \cdot = r^2. \tag{2}$$

In order, therefore, to determine r_x, r_y, . . . we must make two computations, one of the weights p_x, p_y, . . . and the other of r, the p. e. of a single observation.

It is evident from an inspection of the normal equations that x, y, . . . are linear functions of l_1, l_2, . . . Let, then,

$$\begin{aligned} x &= \alpha_1 l_1 + \alpha_2 l_2 + \cdot \cdot \cdot + \alpha_n l_n = [\alpha l], \\ y &= \beta_1 l_1 + \beta_2 l_2 + \cdot \cdot \cdot + \beta_n l_n = [\beta l], \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{3}$$

in which α_1, α_2, . . . ; β_1, β_2, . . . ; . . . are functions of a_1, b_1, . . . ; a_2, b_2, . . . ; . . . their values being as yet undetermined.

Now, r being the p. e. of each of the observed quantities M_1, M_2, . . . M_n, must be also the p. e. of l_1, l_2, . . . l_n, which differ from M_1, M_2, . . . M_n by known amounts (see Art. 74). Hence, since l_1, l_2, . . . l_n are independent of each other (Art. 13),

$$r_x^2 = r^2 [\alpha\alpha],\ r_y^2 = r^2 [\beta\beta],\ . . . \tag{4}$$

and therefore

$$p_x = \frac{1}{[\alpha\alpha]},\quad p_y = \frac{1}{[\beta\beta]},\ . . . \tag{5}$$

We shall first of all determine the weights p_x, p_y, . . .

97. The demonstration may be carried out simply by the application of the principles of undetermined coefficients. Thus, substitute $[\alpha l]$, $[\beta l]$, . . . for x, y, . . . in the normal equations (1), and

$$\begin{aligned} [aa][\alpha l] + [ab][\beta l] + \cdot \cdot \cdot &= [al], \\ [ab][\alpha l] + [bb][\beta l] + \cdot \cdot \cdot &= [bl], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &\quad \cdot \quad \cdot \end{aligned} \tag{6}$$

or, arranging according to l_1, l_2, . . .

$$\begin{aligned} &\{[aa]\alpha_1 + [ab]\beta_1 + \cdot \cdot \cdot - a_1\}l_1 \\ &\qquad + \{[aa]\alpha_2 + [ab]\beta_2 + \cdot \cdot \cdot - a_2\}l_2 + \cdot \cdot \cdot = 0. \\ &\{[ab]\alpha_1 + [bb]\beta_1 + \cdot \cdot \cdot - b_1\}l_1 \\ &\qquad + \{[ab]\alpha_2 + [bb]\beta_2 + \cdot \cdot \cdot - b_2\}l_2 + \cdot \cdot \cdot = 0. \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

The unknown quantities $\alpha_1, \alpha_2, \ldots$ may be so determined that the coefficients of $l_1, l_2, \ldots$ shall each equal zero. Hence the several sets of equations,

$$\left\{\begin{array}{l} [aa]\,\alpha_1 + [ab]\,\beta_1 + \cdots - a_1 = 0, \\ [aa]\,\alpha_2 + [ab]\,\beta_2 + \cdots - a_2 = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right. \qquad (7)$$
$$\left\{\begin{array}{l} [ab]\,\alpha_1 + [bb]\,\beta_1 + \cdots - b_1 = 0, \\ [ab]\,\alpha_2 + [bb]\,\beta_2 + \cdots - b_2 = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

are simultaneously satisfied by the same values of

$$\alpha_1, \alpha_2, \ldots; \ \beta_1, \beta_2, \ldots; \ \ldots$$

Multiply the equations of each set by $a_1, a_2, \ldots$ in order, and add; then necessarily

$$[a\alpha] = 1, \ [a\beta] = 0, \ [a\gamma] = 0, \ \ldots \qquad (8)$$

In a similar way, multiplying by $b_1, b_2, \ldots; c_1, c_2, \ldots$, etc., and adding, there result

$$\begin{array}{l} [b\alpha] = 0, \ [b\beta] = 1, \ [b\gamma] = 0, \ \ldots \\ [c\alpha] = 0, \ [c\beta] = 0, \ [c\gamma] = 1, \ \ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

Again, multiply the first set by $\alpha_1, \alpha_2, \ldots$, the second by $\beta_1, \beta_2, \ldots$, and so on, and add, and we have the sets of equations,

$$\left\{\begin{array}{l} [aa]\,[\alpha\alpha] + [ab]\,[\alpha\beta] + \cdots = [a\alpha] = 1, \\ [ab]\,[\alpha\alpha] + [bb]\,[\alpha\beta] + \cdots = [b\alpha] = 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right. \qquad (9)$$
$$\left\{\begin{array}{l} [aa]\,[\alpha\beta] + [ab]\,[\beta\beta] + \cdots = [a\beta] = 0, \\ [ab]\,[\alpha\beta] + [bb]\,[\beta\beta] + \cdots = [b\beta] = 1, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

from which equations $[\alpha\alpha], [\alpha\beta], \ldots$ may be found.

It is plain that the coefficients of $[\alpha\alpha], [\alpha\beta], \ldots; [\alpha\beta], [\beta\beta], \ldots$ in these equations are the same as those of $x, y,$

. . . in the normal equations, and that the absolute terms are 1, 0, . . . ; 0, 1, . . . ; . . . instead of $[al]$, $[bl]$, . . . Hence,

98. To Find the Weights of the Unknowns. — *In the first normal equation write* 1 *for* $[al]$, *and in the other normal equations put* 0 *for each of* $[bl]$, $[cl]$, . . . ; *the value of* x *found from these equations will be the reciprocal of the weight of* x, *and the values of* y, z, . . . *will be the values of* $[\alpha\beta]$, $[\alpha\gamma]$, . . . *In the second normal equation write* 1 *for* $[bl]$, *and in the other equations put* 0 *for each of* $[al]$, $[cl]$, . . . ; *the value of* y *found from these equations will be the reciprocal of the weight of* y, *and the values of* z, . . . *found will be the values of* $[\beta\gamma]$, . . . Similarly for each of the unknowns in succession. For example, the weight equations for three unknowns are

$[\alpha\alpha]$	$[\alpha\beta]$	$[\alpha\gamma]$	
$+[aa]$	$+[ab]$	$+[ac]$	1
$+[ab]$	$+[bb]$	$+[bc]$	0
$+[ac]$	$+[bc]$	$+[cc]$	0

$[\alpha\beta]$	$[\beta\beta]$	$[\beta\gamma]$	
$+[aa]$	$+[ab]$	$+[ac]$	0
$+[ab]$	$+[bb]$	$+[bc]$	1
$+[ac]$	$+[bc]$	$+[cc]$	0

$[\alpha\gamma]$	$[\beta\gamma]$	$[\gamma\gamma]$	
$+[aa]$	$+[ab]$	$+[ac]$	0
$+[ab]$	$+[bb]$	$+[bc]$	0
$+[ac]$	$+[bc]$	$+[cc]$	1

The quantities $[\alpha\beta]$, $[\alpha\gamma]$, . . . are necessary when the weight of a linear function of the unknowns is required, as will be seen presently. (See Art. 108.)

99. It is evident from the form of the weight equations that if the elimination is carried through by the method of substitution, the successive steps to the left of the sign of equality are the same as in Arts. 86–88. Hence if the equations are arranged so that the unknown whose weight is required — z, for example — is found first, we should have the forms

x	y	z	
$[aa]$	$+[ab]$	$+[ac]$	$[al]$
. . .	$+[bb.1]$	$+[bc.1]$	$[bl.1]$
. . .	. . .	$+[cc.2]$	$[cl.2]$
		$\therefore [cc.2]z = [cl.2]$	

$[\alpha\gamma]$	$[\beta\gamma]$	$[\gamma\gamma]$	
$[aa]$	$+[ab]$	$+[ac]$	0
. . .	$+[bb.1]$	$+[bc.1]$	0
. . .	. . .	$+[cc.2]$	1
		$\therefore [cc.2]\,[\gamma\gamma] = 1$	

Hence the coefficient of the unknown first found in the ordinary solution of the normal equations is the weight of that unknown. By a separate elimination for each unknown, the weight of that unknown could be found as above, but the process would be intolerably tedious.

100. *Special Cases of Two and Three Unknowns.* — We may, however, from the preceding derive formulas for the weights in a series of normal equations containing not more than three unknowns, which are easy of application.

Thus with two unknowns, x and y, y being found first,

$$\begin{aligned} p_y &= [bb.1] \\ &= [bb] - \frac{[ab]^2}{[aa]}. \end{aligned}$$

In the reverse order, x being found first,

$$p_x = [aa] - \frac{[ab]^2}{[bb]},$$

or

$$p_x = \frac{\lambda}{[bb]}, \quad p_y = \frac{\lambda}{[aa]},$$

where

$$\lambda = [aa]\,[bb] - [ab]\,[ab].$$

With three unknowns, x, y, z, performing the elimination of the normal equations in the order z, y, x, we have

$$\begin{aligned} p_z &= [cc.2], \\ p_y &= \frac{[bb.1]\,[cc.2]}{[cc.1]}, \\ p_x &= \frac{[aa]\,[bb.1]\,[cc.2]}{[bb]\,[cc] - [bc]\,[bc]}, \end{aligned}$$

which expressions are easily transformed into

$$\begin{aligned} p_z &= \frac{\lambda}{[aa]\,[bb] - [ab]^2}, \\ p_y &= \frac{\lambda}{[aa]\,[cc] - [ac]^2}, \end{aligned}$$

$$p_x = \frac{\lambda}{[bb]\,[cc] - [bc]^2},$$

where

$$\lambda = [aa]\,[bb]\,[cc] + 2\,[ab]\,[bc]\,[ac] - [aa]\,[bc]^2 - [bb]\,[ac]^2 - [cc]\,[ab]^2.$$

From these formulas the weights of the unknowns can be found directly without solving the normal equations. If the normal equations have simple coefficients, it is much more rapid to find the weights in this way and solve the equations by ordinary algebra rather than by the Gaussian method. But when the number of unknowns exceeds three, this becomes too cumbersome.

Ex. — To find the weights of the adjusted angles in Ex. 4, Art. 77. Here

$$\begin{aligned} \lambda &= 12 \times 11 \times 15 - 12 \times 16 - 11 \times 49 \\ &= 1249 \end{aligned}$$

and

$$p_x = \frac{1249}{149} = 8.4,$$

$$p_y = \frac{1249}{131} = 9.5,$$

$$p_z = \frac{1249}{132} = 9.5.$$

If u_x, u_y, u_z denote the reciprocals of p_x, p_y, p_z respectively, then

$$u_x = 0.1193,$$
$$u_y = 0.1049,$$
$$u_z = 0.1057.$$

101. *Modification of General Method.* — To carry out the method of Art. 98 directly as stated would be excessively troublesome, and various modifications have been proposed. The following scheme, which consists in running the weight equations together, will be found very convenient.

Take, for simplicity in writing, three unknowns, x, y, z, and to the ordinary form of the normal equations as arranged for solution add the columns

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

the check being carried throughout.

Perform the elimination exactly as stated in Art. 86, and find the values of the unknowns in the usual way. We have then

x	y	z		R	S	T	Check.
$[aa]$	$[ab]$	$[ac]$	$[al]$	1	0	0	$[as]+1$
$[ab]$	$[bb]$	$[bc]$	$[bl]$	0	1	0	$[bs]+1$
$[ac]$	$[bc]$	$[cc]$	$[cl]$	0	0	1	$[cs]+1$
1	$\frac{[ab]}{[aa]}$	$\frac{[ac]}{[aa]}$	$\frac{[al]}{[aa]}$	$\frac{1}{[aa]}$	. . .	. . .	. . .
. .	$[bb.1]$	$[bc.1]$	$[bl.1]$	R_1	1	. . .	. . .
.	$[bc.1]$	$[cc.1]$	$[cl.1]$	$-\frac{[ac]}{[aa]}$	0	1	. . .
. .						. . .	. . .
. .	. . .	$\frac{[bc.1]}{[bb.1]}$	$\frac{[bl.1]}{[bb.1]}$			. . .	. . .
. .	1			$\frac{R_1}{[bb.1]}$	$\frac{1}{[bb.1]}$	. . .	. . .
. .	. . .	$[cc.2]$	$[cl.2]$	R_2	S_2	1	. . .
		. . .					
. .	. . .			R_2	S_2	1	. . .
. .	. . .	1	$\frac{[cl.2]}{[cc.2]}$	$\frac{R_2}{[cc.2]}$	$\frac{S_2}{[cc.2]}$	$\frac{1}{[cc.2]}$	. . .

where

$$0 = \frac{[ab]}{[aa]} + R_1,$$

$$0 = \frac{[ac]}{[aa]} + \frac{[bc.1]}{[bb.1]} R_1 + R_2,$$

$$0 = \frac{[bc.1]}{[bb.1]} + S_2.$$

Now, taking the first column in the table under the heading R, and attending to Art. 98, we have

$$[\alpha\gamma] = \frac{R_2}{[cc.2]},$$

$$[\alpha\beta] = \frac{R_1}{[bb.1]} - \frac{[bc.1]}{[bb.1]}[\alpha\gamma]$$

$$= \frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc.2]},$$

$$u_x = [\alpha\alpha] = \frac{1}{[aa]} - \frac{[ab]}{[aa]}[\alpha\beta] - \frac{[ac]}{[aa]}[\alpha\gamma]$$

$$= \frac{1}{[aa]} + \frac{R_1^2}{[bb.1]} + \frac{R_2^2}{[cc.2]}.$$

Similarly for the column under S,

$$[\beta\gamma] = \frac{S_2}{[cc.2]},$$

$$u_y = [\beta\beta] = \frac{1}{[bb.1]} + \frac{S_2^2}{[cc.2]},$$

and for the column under T,

$$u_z = [\gamma\gamma] = \frac{1}{[cc.2]}.$$

Also it is evident that

$$z = \frac{[cl.2]}{[cc.2]},$$

$$y = \frac{[bl.1]}{[bb.1]} + \frac{[cl.2]}{[cc.2]} S_2,$$

$$x = \frac{[al]}{[aa]} + \frac{[bl.1]}{[bb.1]} R_1 + \frac{[cl.2]}{[cc.2]} R_2.$$

The forms of the expressions for $[aa]$, $[\beta\beta]$, $[\gamma\gamma]$, . . . show that these quantities may be conveniently computed from the preceding tabular elimination scheme. Thus the sum of the products of each pair of numbers bracketed under the heads R, S, T will give u_x, u_y, u_z respectively.

The convenience of this form is seen in such a case as the following, which is of common occurrence. In a set of, say, 40 normal equations, the weights of 10 of the unknowns may be required. These 10 would be placed last in the solution of the equations, and the extra columns R, S, . . . added after 30 of the unknowns had been eliminated, thus giving the weights required, with a trifling increase of work.

102. **Ex. 1.** — Given the normal equations

$$\begin{aligned} 12x \qquad\quad - 7z &= R \\ + 11y - 4z &= S \\ -7x - 4y + 15z &= T \end{aligned}$$

to find the weights of y and z.

x	y	z	S	T
+ 12	0	− 7	. . . . . .	. . . .
	+ 11	− 4	. . . . . .	. . . .
− 7	− 4	+ 15	. . . . . .	. . . .
1	0	− 0.5833	. . . . . .	. . . .
. . . .	+ 11	− 4	+ 1	
. . . .	− 4	+ 10.9169		+ 1
. . . .	1	− 0.3636	+ 0.0909	
. . . .	. . . .	+ 9.4625	+ 0.3636	+ 1
. . . .	. . . .	1	+ 0.0384	+ 0.1057

Hence

$$u_z = 1 \times 0.1057 \qquad\qquad\qquad = 0.1057,$$
$$u_y = 1 \times 0.0909 + 0.3636 \times 0.0384 = 0.1049,$$

agreeing with the values in Art. 100.

Ex. 2. — Show that

$$[bl.\ 1] = [al]\, R_1 + [bl].$$
$$[cl.\ 2] = [al]\, R_2 + [bl]\, S_2 + [cl],$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Ex. 3. — Show that the multipliers R_1, R_2, . . . satisfy the conditions

$$[aa]\, R_1 + [ab] = 0,$$
$$[aa]\, R_2 + [ab]\, S_2 + [ac] = 0,$$
$$[ab]\, R_2 + [bb]\, S_2 + [bc] = 0,$$
$$[aa]\, R_3 + [ab]\, S_3 + [ac]\, T_3 + [ad] = 0,$$
$$[ab]\, R_3 + [bb]\, S_3 + [bc]\, T_3 + [bd] = 0,$$
$$[ac]\, R_3 + [bc]\, S_3 + [cc]\, T_3 + [cd] = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

103. Second Method of Finding the Weights of the Unknowns. — If we multiply the first of the normal equations 1, Art. 96, by $[\alpha\alpha]$, the second by $[\alpha\beta]$, the third by $[\alpha\gamma]$, and so on; add the products, and attend to equations 9, Art. 97, we obtain

$$x = [\alpha\alpha]\,[al] + [\alpha\beta]\,[bl] + [\alpha\gamma]\,[cl] + \cdot \cdot \cdot$$

Similarly

$$y = [\alpha\beta]\,[al] + [\beta\beta]\,[bl] + [\beta\gamma]\,[cl] + \cdot \cdot \cdot$$
$$z = [\alpha\gamma]\,[al] + [\beta\gamma]\,[bl] + [\gamma\gamma]\,[cl] + \cdot \cdot \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence, since $[\alpha\alpha]$, $[\beta\beta]$, . . . are the reciprocals of the weights of x, y, . . ., this method of finding the weights may be stated as follows:

In any given series of observations, having formed the normal equations, replace the numerical absolute terms by the general symbols $[al]$, $[bl]$, . . . *and find by any method of elimination the values of* x, y, . . ., *in terms of* $[al]$, $[bl]$, . . .; *then the weight of* x *is the reciprocal of the coefficient of* $[al]$ *in the value of* x, *the weight of* y *is the reciprocal of the coefficient of* $[bl]$ *in the value of* y, *and so on.*

The coefficients of the remaining symbols for the absolute terms in the expressions for x, y, . . . give the values of $[\alpha\beta]$, $[\alpha\gamma]$, . . . ; $[\beta\gamma]$, . . . ; . . . and the numerical values of the unknowns x, y, . . . may be found by substituting for $[al]$, $[bl]$, . . . their numerical values.

In this method of computing the values of the unknowns and their weights, a machine can be used with great advantage.

The formulas of Art. 101 are easily derived from the preceding principles. For solving the normal equations

$$\begin{aligned}[aa]\,x + [ab]\,y + [ac]\,z &= [al],\\ [ab]\,x + [bb]\,y + [bc]\,z &= [bl],\\ [ac]\,x + [bc]\,y + [cc]\,z &= [cl],\end{aligned}$$

by the method of substitution we have for the first unknown,

$$\begin{aligned}x &= \frac{[al]}{[aa]} + \frac{[bl.1]}{[bb.1]}R_1 + \frac{[cl.2]}{[cc.2]}R_2\\ &= [al]\left(\frac{1}{[aa]} + \frac{R_1^2}{[bb.1]} + \frac{R_2^2}{[cc.2]}\right) + [bl]\left(\frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc.2]}\right) + [cl]\frac{R_2}{[cc.2]}.\end{aligned}$$

Comparing this with the general expression for x in Eq. 1,

$$\begin{aligned}[\alpha\alpha] &= \frac{1}{[aa]} + \frac{R_1^2}{[bb.1]} + \frac{R_2^2}{[cc.2]},\\ [\alpha\beta] &= \frac{R_1}{[bb.1]} + \frac{R_2 S_2}{[cc.2]},\\ [\alpha\gamma] &= \frac{R_2}{[cc.2]}.\end{aligned}$$

Similarly for y and z.

Ex. — To find the weights of the unknowns in Ex. 4, Art. 77. Solving the normal equations in general terms,

$$\begin{aligned} x &= 0.1193\,[al] + 0.0224\,[bl] + 0.0616\,[cl], \\ y &= 0.0224\,[al] + 0.1049\,[bl] + 0.0384\,[cl], \\ z &= 0.0616\,[al] + 0.0384\,[bl] + 0.1057\,[cl]. \end{aligned}$$

Hence

$$\begin{aligned} u_x &= [\alpha\alpha] = 0.1193, \\ u_y &= [\beta\beta] = 0.1049, \\ u_z &= [\gamma\gamma] = 0.1057, \end{aligned}$$

as found in Art. 100.

104. In deducing the formulas for the precision of the adjusted values in a series of normal equations, we have, for convenience in writing, taken the observation equations to be reduced to weight unity, and the normal equations, consequently, to be of the form

$$\begin{aligned} [aa]\,x + [ab]\,y + \cdots &= [al], \\ [ab]\,x + [bb]\,y + \cdots &= [bl], \\ \cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (1)$$

The formulas with the weight symbols introduced, corresponding to those found in the preceding articles, are easily derived from them by writing $a\sqrt{p}$, $b\sqrt{p}$, ... $l\sqrt{p}$, for a, b, ... l, and $\alpha\sqrt{u}$, $\beta\sqrt{u}$, ... for α, β, ... respectively. (See Art. 48.)

Thus, for example, from the normal equations

$$\begin{aligned} [paa]\,x + [pab]\,y + \cdots &= [pal], \\ [pab]\,x + [pbb]\,y + \cdots &= [pbl], \\ \cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (2)$$

we should have

$$\begin{aligned} x = [\alpha l] &= [u\alpha\alpha][pal] + [u\alpha\beta][pbl] + \cdots \\ y = [\beta l] &= [u\alpha\beta][pal] + [u\beta\beta][pbl] + \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (3)$$

and by equating coefficients of l_1, l_2, ... in the first expression,

$$\begin{aligned} [u\alpha\alpha]\,a_1 + [u\alpha\beta]\,b_1 + \cdots &= u_1\alpha_1, \\ [u\alpha\beta]\,a_1 + [u\beta\beta]\,b_1 + \cdots &= u_1\beta_1. \\ \cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (4)$$

105. To Find the p. e. r of a Single Observation. — If the errors Δ were known — that is, if the n observation equations were

$$\begin{aligned} a_1x_0 + b_1y_0 + \cdots - l_1 &= \Delta_1, \\ a_2x_0 + b_2y_0 + \cdots - l_2 &= \Delta_2, \\ \cdots\cdots\cdots\cdots \end{aligned} \tag{1}$$

where x_0, y_0, . . . are the true values of the unknowns — we should have at once

$$\mu^2 = \frac{[\Delta^2]}{n}.$$

But we have only the residuals v with the observation equations

$$\begin{aligned} a_1x + b_1y + \cdots - l_1 &= v_1, \\ a_2x + b_2y + \cdots - l_2 &= v_2, \\ \cdots\cdots\cdots\cdots \end{aligned} \tag{2}$$

where x, y, . . . are the most probable values of the unknowns. We must, therefore, express $[\Delta^2]$ in terms of the residuals v in order to find μ.

From the two sets of equations, by subtracting in pairs,

$$\begin{aligned} \Delta_1 &= v_1 + a_1(x_0 - x) + b_1(y_0 - y) + \cdots \\ \Delta_2 &= v_2 + a_2(x_0 - x) + b_2(y_0 - y) + \cdots \\ \cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

Now, taking the m. s. e. μ_x, μ_y, . . . to be the errors of x, y, . . . , that is, to be equal to $x_0 - x$, $y_0 - y$, . . . , we have from Eq. 4, Art. 96,

$$x_0 - x = \mu\sqrt{[\alpha\alpha]}, \quad y_0 - y = \mu\sqrt{[\beta\beta]}, \ldots$$

and therefore

$$\begin{aligned} \Delta_1 &= v_1 + \mu(a_1\sqrt{[\alpha\alpha]} + b_1\sqrt{[\beta\beta]} + \cdots), \\ \Delta_2 &= v_2 + \mu(a_2\sqrt{[\alpha\alpha]} + b_2\sqrt{[\beta\beta]} + \cdots). \\ \cdots\cdots\cdots\cdots \end{aligned}$$

Squaring, adding, and attending to equations (9), Art. 97, we have approximately, n_i being the number of unknowns,

$$[\Delta^2] = [v^2] + n_i\mu^2. \tag{4}$$

Putting $[\Delta^2] = n\mu^2$, there results

$$\mu^2 = \frac{[v^2]}{n - n_i} \text{ and } r = .6745\sqrt{\frac{[v^2]}{n - n_i}}, \tag{5}$$

the expression required.

Reasoning as in Peters' formula, Art. 32, we easily deduce from (4)

$$r = 0.8653 \frac{[v}{\sqrt{n\,(n - n_i)}}, \tag{6}$$

which is known as Lüroth's formula (*Astron. Nachr.*, 1740).

When $n_i = 1$, equations (5) and (6) reduce to Bessel's and Peters' formulas respectively (Arts. 29, 32).

106. *Methods of Computing* $[v^2]$. — (*a*) The ordinary method is to substitute the values of the unknowns found from the solution of the normal equations in the observation equations, and thence find v_1, v_2, . . . The sum of the squares of these residuals will give $[v^2]$.

The residuals having to be found, for the purpose of testing the quality of the work this method of computing $[v^2]$ is on the whole as short as any.

As checks on the values of $[v^2]$ found in this way the following are of value :

(*b*) If we multiply each observation equation by its v and take the sum of the products, then, remembering that $[av] = 0$, $[bv] = 0$, . . . , we find

$$[v^2] = -[vl].$$

(*c*) If we multiply each of the observation equations by its l and take the sum of the products,

$$[al]\,x + [bl]\,y + \cdots - [ll] = [vl] = -[v^2].$$

(*d*) We have for two unknowns, x and y,

$$\begin{aligned}[v^2] &= [(ax + by - l)^2] \\ &= [aa]\,x^2 + 2\,[ab]\,xy + [bb]\,y^2 - 2\,[al]\,x - 2\,[bl]\,y + [ll] \\ &= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + \left([bb] - \frac{[ab]}{[aa]}\,[ab]\right)y^2\end{aligned}$$

$$
\begin{aligned}
&- \left(2\,[bl] - 2\,\frac{[ab]}{[aa]}[al]\right)y + [ll] - \frac{[al]}{[aa]}[al] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + [bb.1]y^2 - 2\,[bl.1]y + [ll.1] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + [bb.1]\left(y - \frac{[bl.1]}{[bb.1]}\right)^2 \\
&\qquad + [ll.1] - \frac{[bl.1]}{[bb.1]}[bl.1] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y - \frac{[al]}{[aa]}\right)^2 + [bb.1]\left(y - \frac{[bl.1]}{[bb.1]}\right)^2 + [ll.2],
\end{aligned}
$$

and generally for m unknowns,

$$
\begin{aligned}
[v^2] &= [(ax + by + \cdots - l)^2] \\
&= [aa]\left(x + \frac{[ab]}{[aa]}y + \cdots - \frac{[al]}{[aa]}\right)^2 \\
&\quad + [bb.1]\left(y + \frac{[bc.1]}{[bb.1]}z + \cdots - \frac{[bl.1]}{[bb.1]}\right)^2 + \cdots + [ll.m].
\end{aligned}
$$

Now, from (9), Art. 85, the coefficients of $[aa]$, $[bb.1]$, . . . are each equal to zero. Hence

$$
\begin{aligned}
[vv] &= [ll.m] \\
&= [ll] - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \frac{[cl.2]^2}{[cc.2]} - \cdots
\end{aligned}
$$

This expression was first given by Gauss (*De Elementis Ellipticis Palladis*, Art. 13). Its form suggests that if we add an extra column to the normal equations, as shown in the following scheme, we shall find $[v^2]$ at the same time as the first unknown. This is analytically very elegant, and, as the check (see Art. 85) can be carried with this column through the solution of the normal equations, it may be used for finding $[v^2]$, if one is computing alone. Only one extra term $[ll]$ has to be computed while forming the normal equations.

The scheme is as follows:

x	y	z	
$[aa]$	$+[ab]$	$+[ac]$	$[al]$
..	$+[bb]$	$+[bc]$	$[bl]$
..	...	$+[cc]$	$[cl]$
..	...	...	$[ll]$
1	$+\frac{[ab]}{[aa]}$	$+\frac{[ac]}{[aa]}$	$\frac{[al]}{[aa]}$
..	$+[bb.1]$	$+[bc.1]$	$[bl.1]$
..	...	$+[cc.1]$	$[cl.1]$
..	...	...	$[ll.1]=[ll]-\frac{[al]}{[aa]}[al]$
..	1	$+\frac{[bc.1]}{[bb.1]}$	$\frac{[bl.1]}{[bb.1]}$
..	..	$[cc.2]$	$[cl.2]$
..	..	...	$[ll.2]=[ll.1]-\frac{[bl.1]}{[bb.1]}[bl.1]$

107. Ex. 1. — To find the m. s. e. of the adjusted values of the unknowns found in Ex. 4, Art. 77.

The first step is to find $[pv^2]$. This we shall do in the four ways indicated.

(*a*)

v	p	pv^2
-0.05	5	0.01
-0.36	7	0.91
$+0.68$	4	1.85
-0.03	7	0.01
-0.62	4	1.54
		$4.32=[pv^2]$

(*b*)

$$p_1l_1v_1=0$$
$$p_2l_2v_2=0$$
$$p_3l_3v_3=0$$
$$p_4l_4v_4=7\times 0.76\times -0.03=-0.16$$
$$p_5l_5v_5=4\times 1.66\times -0.62=-4.12$$
$$-4.28=+[pvl]=-[pv^2]$$

(*c*)

$$
\begin{array}{lll}
[pal]\,x = -5.32 \times -0.05 = 0.27 & & \\
[pbl]\,y = -6.64 \times -0.36 = 2.39 & & \\
[pcl]\,z = +11.96 \times +0.68 = 8.13 & & \\
 & & 10.79 \\
p_1 l_1 l_1 = 0 & & \\
p_2 l_2 l_2 = 0 & & \\
p_3 l_3 l_3 = 0 & & \\
p_4 l_4 l_4 = 7 \times (0.76)^2 & = 4.04 & \\
p_5 l_5 l_5 = 4 \times (1.66)^2 & = 11.02 & \\
 & & 15.06 \\
 & & 4.27 = [pv^2]
\end{array}
$$

(*d*) We find $[pll] = 15.06$.

The solution of the normal equations, with the extra column for $[pll]$ added, would be, according to the foregoing scheme,

x	y	z		
12	0	− 7	− 5.32	
. .	+ 11	− 4	− 6.64	
. .	. . .	+ 15	+ 11.96	
. .	. . .	. . .	+ 15.06	2.36
1	0	− 0.583	− 0.443	
. .	+ 11	− 4	− 6.64	
. .	. . .	+ 10.917	+ 8.857	
. .	. . .	. . .	+ 12.70	(= 15.06 − 2.36)
	1	− 0.364	− 0.604	
		+ 9.462	+ 6.422	
		. . .	+ 8.69	(= 12.70 − 4.01)
		1	+ 0.68	
		$[pvv] =$	+ 4.30	(= 8.69 − 4.39)

Mean value of $[pv^2] = 4.29$.
Hence (Art. 105)

$$\mu = \sqrt{\frac{4.29}{5-3}} = 1.47''$$

and (see Ex., Art. 103)

$$\mu_x = 1.47''\sqrt{0.1193}, \quad \mu_y = 1.47''\sqrt{0.1049}, \quad \mu_z = 1.47''\sqrt{0.1057}$$
$$= 0.51'' \qquad\qquad = 0.48'' \qquad\qquad = 0.48''$$

Ex. 2. — Show that

$$[\Delta v] = [v^2].$$

[Multiply equations 1, Art. 105, by $v_1, v_2, \ldots$ and add. Then, since,

$$[av] = 0, \quad [bv] = 0, \quad \ldots$$

$$\therefore [\Delta v] = -[lv]\ ].$$

Ex. 3. — Show that

$$[v^2] = [\Delta\Delta] - \frac{[a\Delta]^2}{[aa]} - \frac{[b\Delta.1]^2}{[bb.1]} - \ldots$$

[Form the normal equations from equations (3), Art. 105, and

$$[aa](x_0 - x) + [ab](y_0 - y) + \ldots = [a\Delta],$$

$$[ab](x_0 - x) + [bb](y_0 - y) + \ldots = [b\Delta],$$

$$\cdots\cdots\cdots\cdots$$

since $\qquad [av] = [bv] = \ldots = 0.$

Hence, from Art. 106,

$$[\Delta^2] = [v^2] + \frac{[a\Delta]^2}{[aa]} + \frac{[b\Delta.1]^2}{[bb.1]} + \ldots]$$

Ex. 4. — From the equation

$$[al]x + [bl]y + \ldots - [ll] = -[v^2],$$

and

$$x = \frac{[al]}{[aa]} + \frac{[bl.1]}{[bb.1]}R_1 + \frac{[cl.2]}{[cc.2]}R_2 + \ldots$$

$$y = \qquad \frac{[bl.1]}{[bb.1]} \qquad + \frac{[cl.2]}{[cc.2]}S_2 + \ldots$$

$$\cdots\cdots\cdots\cdots$$

deduce

$$[v^2] = [ll] - \frac{[al]^2}{[aa]} - \frac{[bl.1]^2}{[bb.1]} - \ldots$$

Ex. 5. — Prove that $\quad [\alpha v] = [\beta v] = \ldots = 0.$

Ex. 6. — From

$$[v^2] = [\Delta^2] - \frac{[a\Delta]^2}{[aa]} - \frac{[b\Delta.1]^2}{[bb.1]} - \ldots$$

deduce the formula

$$\mu^2 = \frac{[v^2]}{n - n_i}.$$

108. To Find the Precision of any Function of the Adjusted Values $X, Y, \ldots$ — This is the more general case of the problem just discussed. The method of solution is:

First, to find r, the p. e. of an observation of weight unity, and next p_F the weight of the function, whence the p. e. of the function is given by

$$r\sqrt{u_F},$$

where u_F is the reciprocal of p_F.

The value of r is computed from (5) or (6), Art. 105.

Next, to find u_F. Let the function be

$$F = f(X, Y, \ldots) \tag{1}$$

in which $X, Y, \ldots$ are functions of the independently observed quantities $M_1, M_2, \ldots M_n$.

Reducing the function to the linear form, we have, adopting the notation of Art. 74,

$$\begin{aligned} F &= f(X' + x, Y' + y, \ldots) \\ &= f(X', Y', \ldots) + \frac{\delta F}{\delta X'} x + \frac{\delta F}{\delta Y'} y + \cdots \end{aligned} \tag{2}$$

or, as it may be written,

$$dF = G_1 x + G_2 y + \cdots \tag{3}$$

Now, since $x, y, \ldots$ are not independent, but are connected by the equations

$$\begin{aligned} [aa]x + [ab]y + \cdots &= [al], \\ [ab]x + [bb]y + \cdots &= [bl], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \end{aligned}$$

we must get rid of this entanglement by expressing these quantities $x, y, \ldots$ in terms of $l_1, l_2, \ldots$, which are independent of each other. From Arts. 96–97 we may write

$$x = [\alpha l], \quad y = [\beta l], \ldots$$

where $\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots; \ldots$ are functions of $a_1, b_1, \ldots; a_2, b_2, \ldots; \ldots$ Hence, substituting in (3),

$$dF = (G_1\alpha_1 + G_2\beta_1 + \cdots) l_1 + (G_1\alpha_2 + G_2\beta_2 + \cdots) l_2 + \cdots$$

and, therefore (Arts. 13, 47), since the observation equations have been reduced to weight unity,

$$\begin{aligned} u_F &= (G_1\alpha_1 + G_2\beta_1 + \cdots)^2 + (G_1\alpha_2 + G_2\beta_2 + \cdots)^2 + \cdots \\ &= G_1^2 [\alpha\alpha] + 2\, G_1 G_2 [\alpha\beta] + \cdots \\ &\qquad + G_2^2 \quad [\beta\beta] + \cdots \\ &\qquad\qquad + \cdots \end{aligned} \tag{4}$$

where $[\alpha\alpha]$, $[\alpha\beta] \ldots$ may be found in the manner indicated in Arts. 98, 101, or 103. Hence u_F is known.

109. (*b*) Eq. 4 may be written

$$u_F = G_1 Q_1 + G_2 Q_2 + \cdots + G_n Q_n, \tag{5}$$

where

$$\begin{aligned} Q_1 &= [\alpha\alpha]\, G_1 + [\alpha\beta]\, G_2 + \cdots \\ Q_2 &= [\alpha\beta]\, G_1 + [\beta\beta]\, G_2 + \cdots \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{6}$$

that is, where (see Eq. 1, Art. 103)

$$\begin{aligned} G_1 &= [aa]\, Q_1 + [ab]\, Q_1 + \cdots \\ G_2 &= [ab]\, Q_2 + [bb]\, Q_2 + \cdots \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{7}$$

Hence the weight of a function

$$G_1 x + G_2 y + \cdots$$

of several independent unknowns $x, y, \ldots$ is found from

$$u_F = [GQ],$$

where $Q_1, Q_2, \ldots$ satisfy the equations

$$\begin{aligned} [aa]\, Q_1 + [ab]\, Q_2 + \cdots &= G_1, \\ [ab]\, Q_1 + [bb]\, Q_2 + \cdots &= G_2, \\ \cdots\cdots\cdots\cdots \end{aligned}$$

Therefore, we conclude that, *if in a series of observation equations the values of the unknowns $x, y, \ldots$ are required, as well as their weights or the weight of any function of them, these results can be found at one time by making a solution of the normal equations for finding $x, y, \ldots$ in general terms, and then substituting for $[al]$, $[bl]$, $\ldots$ their numerical values on the one hand and the values of $G_1, G_2, \ldots$ on the other.*

110. (*c*) This result may be stated in other forms. Thus, from Eq. 4, by substituting for $[\alpha\alpha]$, $[\alpha\beta]$, $\ldots$ their values from Art. 101 or by substituting for $Q_1, Q_2, \ldots$ their values in (6) as expressed in Art. 101 we have, after a simple reduction,

$$u_F = \frac{G_1^2}{[aa]} + \frac{(G_1 R_1 + G_2)^2}{[bb.1]} + \frac{(G_1 R_2 + G_2 S_2 + G_3)^2}{[cc.2]} + \cdots \tag{8}$$

Comparing this expression with (*d*), Art. 106, it is evident that

the several terms are such as would result from the following elimination (Ex. three unknowns) by finding the products of the quantities bracketed :

x	y	z	
$[aa]$ $[ab]$ $[ac]$	$[ab]$ $[bb]$ $[bc]$	$[ac]$ $[bc]$ $[cc]$	G_1 G_2 G_3
1	$\frac{[ab]}{[aa]}$ $[bb.1]$ $[bc.1]$	$\frac{[ac]}{[aa]}$ $[bc.1]$ $[cc.1]$	$\frac{G_1}{[aa]}$ $G_1R_1 + G_2$
	1	$\frac{[bc.1]}{[bb.1]}$ $[cc.2]$	$\frac{G_1R_1 + G_2}{[bb.1]}$ $G_1R_2 + G_2S_2 + G_3$
		1	$\frac{G_1R_2 + G_2S_2 + G_3}{[cc.2]}$

111. (*d*) The expression (8) for u_F may be easily transformed into

$$k_0^2\,[aa] + k_1^2\,[bb.1] + k_2^2\,[cc.2] + \cdots$$

where

$$\begin{aligned}
&[aa]\,k_0 &&= -\,G_1,\\
&[ab]\,k_0 + [bb.1]\,k_1 &&= -\,G_2,\\
&[ac]\,k_0 + [bc.1]\,k_1 + [cc.2]\,k_2 &&= -\,G_3,\\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Circumstances must decide which of the four forms given should be chosen in any special case. A machine can be used to the best advantage with the second and third forms. The third form is also convenient when the weights alone are required, without the values of the unknowns, and the second when the values of the unknowns can be found by an easier method of solution than the Gaussian method of substitution.

112. **Ex. 1.**—In Ex. 4, Art. 77, it is required to find the m. s. e. of the angle PSB.

The function is

$$dF = -x + z.$$

First Solution. From equation (4),

$$\begin{aligned} u_F &= [\alpha\alpha] - 2\,[\alpha\gamma] + [\gamma\gamma] \\ &= 0.1193 - 2 \times 0.0616 + 0.1057 \text{ (from Ex., Art. 103)} \\ &= 0.1018 \end{aligned}$$

Second Solution. From equations (7),

$$\begin{aligned} +12\,Q_1 \qquad\qquad\quad - 7\,Q_3 &= -1 \\ +11\,Q_2 - 4\,Q_3 &= 0 \\ -7\,Q_1 - 4\,Q_2 + 15\,Q_3 &= +1 \end{aligned}$$

Hence

$$Q_1 = -0.0577, \quad Q_3 = +0.0440$$

and

$$\begin{aligned} u_F &= -1 \times -0.0577 + 1 \times 0.0440 \\ &= 0.1017 \end{aligned}$$

Third Solution. Add the extra column G to the solution of the normal equations, which would give the scheme

x	y	z	G
$+12$	. . .	-7	-1
. . .	$+11$	-4	0
. . .	. . .	$+15$	$+1$
$+1$	. . .	-0.5833	-0.0833
. . .	$+11$	$-4.$	0
. . .	. . .	$+10.9169$	$+0.4169$
	$+1$	-0.3636	0
	. . .	$+9.4625$	$+0.4169$
		$+1$	$+0.0441$

Hence

$$\begin{aligned} u_F &= -1 \times -0.0833 + 0.4169 \times 0.0441 \\ &= 0.1017 \end{aligned}$$

Fourth Solution.

$$\begin{aligned} 12\,k_0 \qquad\qquad\qquad\qquad &= +1 \\ 11\,k_1 \qquad\qquad &= 0 \\ -7\,k_0 - 4\,k_1 + 9.4625\,k_2 &= -1 \end{aligned}$$

$$\therefore k_0 = 0.0833,\ k_1 = 0,\ k_2 = -0.0440$$
$$u_F = (0.0833)^2 \times 12 + (0.0440)^2 \times 9.4625$$
$$= 0.1016$$

Also
$$\mu_F{}^2 = 1.47'' \sqrt{0.102}$$
$$= 0.47''.$$

Ex. 2. — Given the observation equations

$$a_1x + b_1y = l_1,$$
$$a_2x + b_2y = l_2,$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$a_nx + b_ny = l_n,$$

to find the weight of $fx + gy$.

[The normal equations are

$$[aa]x + [ab]y = [al],$$
$$[ab]x + [bb]y = [bl].$$

$$\therefore u_F = \frac{1}{[aa]\,[bb.1]}\left\{[bb]f^2 - 2\,[ab]\,[fg] + [aa]g^2\right\}.]$$

113. *To find the average value of the ratio of the weight of the observed value of a quantity to that of its adjusted value in a system of independently observed quantities.*

The adjusted value of the first observed quantity M_1 is $M_1 + v_1$. From Art. 74 it follows that the weight of $M_1 + v_1$ is the same as the weight of $l_1 + v_1$. Now,

$$l_1 + v_1 = a_1x + b_1y + \cdots \tag{1}$$

Hence if P_1 is the weight of the adjusted value $M_1 + v_1$, that is, is the weight of the function $a_1x + b_1y + \ldots$, and $p_1, p_2 \ldots$ are the weights of $l_1, l_2, \ldots$, we have

$$\frac{1}{P_1} = a_1Q_1 + b_1Q_2 + \cdots \tag{2}$$

where (see Eq. 4, Arts. 104, 109)

$$Q_1 = [u\alpha\alpha]\,a_1 + [u\alpha\beta]\,b_1 + \cdots = u_1\alpha_1,$$
$$Q_2 = [u\alpha\beta]\,a_1 + [u\beta\beta]\,b_1 + \cdots = u_1\beta_1. \tag{3}$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

Therefore by substitution of $Q_1, Q_2, \ldots$ in (2),

$$\frac{p_1}{P_1} = a_1\alpha_1 + b_1\beta_1 + \cdots$$

Similarly

$$\frac{p_2}{P_2} = a_2\alpha_2 + b_2\beta_2 + \cdots$$

$$\cdots\cdots\cdots\cdots$$

Hence by addition

$$\left[\frac{p}{P}\right] = [a\alpha] + [b\beta] + \cdots \text{ to } n_i \text{ terms}$$

$$= n_i, \text{ the number of independent unknowns.}$$

Hence the average value of $[p/P]$ is n_i/n, or in words, *the average value of the ratio of the weight of the observed value of the quantity to that of its adjusted value is the ratio of the number of independent unknowns to the number of observed quantities.*

This result may be very readily derived directly as follows: In (1) put $[\alpha l]$ for x, $[\beta l]$ for y, . . . , and we have

$$l_1 + v_1 = (a_1\alpha_1 + b_1\beta_1 + \cdots)\, l_1 + (a_1\alpha_2 + b_1\beta_2 + \cdots)\, l_2 + \cdots$$

Hence, since l_1, l_2, . . . are independent,

$$\begin{aligned}\frac{1}{P_1} &= (a_1\alpha_1 + b_1\beta_1 + \cdots)^2 \frac{1}{p_1} + (a_1\alpha_2 + b_1\beta_2 + \cdots)^2 \frac{1}{p_2} + \cdots \\ &= a_1 \{[u\alpha\alpha]\, a_1 + [u\alpha\beta]\, b_1 + \cdots\} \\ &\quad + b_1 \{[u\alpha\beta]\, a_1 + [u\beta\beta]\, b_1 + \cdots\} \\ &\quad \cdots\cdots\cdots\cdots \\ &= u_1 (a_1\alpha_1 + b_1\beta_1 + \cdots) \qquad (4)\end{aligned}$$

as before.

Ex. — To check the weights of the adjusted values of the angles found in Ex. 4, Art. 77.

The weights of the measured values of the angles are

$$5, \quad 7, \quad 4, \quad 7, \quad 4$$

The weights of the adjusted values are (Ex., Art. 100 and Art. 112),

$$8.4, \quad 9.5, \quad 9.5, \quad 9.8, \quad 7.5$$

Also

$$\begin{aligned}\frac{5}{8.4} + \frac{7}{9.5} + \frac{4}{9.5} + \frac{7}{9.8} + \frac{4}{7.5} &= 3 \\ &= \text{the number of independent unknowns,}\end{aligned}$$

as it should.

Two Special Artifices.

114. The labor of solving and finding the values of the unknowns may be often shortened by taking advantage of some principle inherent in the form of the observation equations themselves. For example, if we have a series of observation equations containing two unknowns, and of which the coefficient of the first unknown is unity, instead of solving in the usual way we may reduce the observation equations to equations containing the second unknown only, and thus solve more readily.

Given

$$x + b_1 y = l_1, \text{ weight } p_1,$$
$$x + b_2 y = l_2, \text{ weight } p_2.$$

Forming the normal equations in the usual way, we have

$$[p] \; x + [pb] \; y = [pl],$$
$$[pb] x + [pb^2] y = [pbl],$$

whence eliminating x,

$$y \{[pb]^2 - [p][pb^2]\} = [pl][pb] - [p][pbl].$$

Now, if the first normal equation is divided by $[p]$, so that

$$x + \frac{[pb]}{[p]} y = \frac{[pl]}{[p]},$$

and from this equation each of the observation equations in succession is subtracted, there result the equations,

$$\left(\frac{[pb]}{[p]} - b_1\right) y = \frac{[pl]}{[p]} - l_1, \text{ weight } p_1,$$
$$\left(\frac{[pb]}{[p]} - b_2\right) y = \frac{[pl]}{[p]} - l_2, \text{ weight } p_2,$$

. . . .

The normal equation for finding y from these equations is,

$$\{[p][pb^2] - [pb^2]\} \; y = [p][pbl] - [pb][pl],$$

the same as results from the elimination of x in the normal equations.

This process is specially convenient if the original observation equations are numerous, and the coefficients b_1, b_2, . . . and the terms l_1, l_2, . . . are large and not widely different.

Ex. — To solve the equations in Ex. 3, Art. 77.
The mean of the equations is

$$x - 0.97\,y = -\,1.97.$$

Subtract each equation from this mean equation, and

$$\begin{aligned} +\,0.62\,y &= +\,0.01, \\ +\,0.47\,y &= -\,0.07, \\ +\,0.26\,y &= +\,0.03, \\ -\,0.01\,y &= -\,0.02, \\ -\,0.25\,y &= +\,0.03, \\ -\,1.08\,y &= 0.00. \end{aligned}$$

The normal equation formed from these equations is

$$+\,1.91\,y = -\,0.27,$$

and $$\therefore\ y = -\,0.014^s.$$

Substitute for y its value in the mean equation, and

$$x = -\,1.95^s.$$

115. Again, we may take advantage of the presence in the problem of some arbitrary quantity to which a convenient value may be assigned. Thus, to find the difference of the coefficients of expansion of two standards A and B from observed differences of length at certain fixed temperatures.

Let x = the excess in length of A over B at an arbitrary temperature t_0,

y = the excess of the coefficient of expansion of A over that of B,

$l_1, l_2, \ldots$ = the observed differences in length at temperatures t_1, t_2, . . . and whose weights are p_1, p_2, . . .

We have then the observation equations

$$\begin{aligned} x + (t_1 - t_0)\,y - l_1 &= v_1, \text{ weight } p_1, \\ x + (t_2 - t_0)\,y - l_2 &= v_2, \text{ weight } p_2, \\ \cdots\cdots&\cdots\cdots \end{aligned} \qquad (1)$$

and the normal equations

$$\begin{aligned} [p]x + \{[pt] - [p]t_0\}\,y &= [pl] \\ \{[pt] - [p]t_0\}\,x + [p(t - t_0)^2]\,y &= [(t - t_0)\,pl]. \end{aligned} \qquad (2)$$

As the value of t_0 is arbitrary, the normal equations will be simplified by taking it equal to the weighted mean of the temperatures; that is,

$$t_0 = \frac{[pt]}{[p]}; \qquad (3)$$

and they will then become

$$\begin{aligned} [p]x &= [pl] \\ [p(t - t_0)^2]\,y &= [(t - t_0)\,pl], \end{aligned} \qquad (4)$$

from which x and y are found at once, with their weights at the mean temperature t_0.

If the values of l are numerically large, it will lessen the labor of finding the value of y if the mean value of x found from

$$[p]\,x = [pl]$$

is substituted in the observation equations before the normal equation in y is formed. We should then have

$$[p(t - t_0)^2]\,y = [p(t - t_0)(l - x)],$$

from which to find y.

It is evident that the value of y found in this way is the same as before. For

$$\begin{aligned} [p(t - t_0)(l - x)] &= [p(t - t_0)\,l] - \{[pt] - [p]t_0\}\,x \\ &= [p(t - t_0)\,l], \end{aligned}$$

since the coefficient of x is equal to zero from Eq. 3.

The quantity $l - x$ comes in very conveniently in computing the residuals v in finding the precision.

The Precision.—If n is the number of observations, the number of unknowns being 2, we have for the m. s. e. μ of a single observation,

$$\mu = \sqrt{\frac{[pvv]}{n-2}},$$

and

$$\mu_x = \frac{\mu}{\sqrt{[p]}},$$

$$\mu_y = \frac{\mu}{\sqrt{[p(t-t_0)^2]}}$$

The length at any temperature t' is

$$x + (t' - t_0)\,y,$$

and its m. s. e. $\mu_{x'}$ is found from

$$\mu_{x'}^2 = \mu_x^2 + (t' - t_0)^2\,\mu_y^2$$
$$= \frac{\mu^2}{[p]} + \frac{(t' - t_0)^2}{[p(t - t_0)^2]}\mu^2.$$

The weight is greatest when μ_x is least, that is, when

$$t' = t_0 = \frac{[pt]}{[p]}.$$

116. Ex.—The following were among the observations made for the determination of the difference of length between the Lake Survey Standard Bar and Yard; and also for the difference between their coefficients of expansion. The unit is $\frac{1}{100000}$ inch.

Required the difference of length at 62° Fahr. and at any other temperature t.

Date.	Observed Temp. (t).	Bar — Yard (l).	Weight (p)
	°		
1872, March 5	24.7	791	5
" 14	37.1	811	1
" 26	61.7	833	6
April 4	49.3	820	6
" 12	66.8	847	8
" 20	71.5	849	8

Let x = the most probable difference between Bar and Yard at 62° Fahr

y = the most probable difference between coefficients of expansion of Bar and Yard.

The observation equations will be of the form

$$x + (t - 62)\,y - l = v.$$

The computation is arranged in tabular form as follows:

p	pt	pl	$t - t_0$	$l - x$
5	123.5	3955	− 32.2	− 40
1	37.1	811	− 19.8	− 20
6	370.2	4998	+ 4.8	+ 2
6	295.8	4920	− 7.6	− 11
8	534.4	6776	+ 9.9	+ 16
8	572.0	6792	+ 14.6	+ 18
34	1933.0	28,252	....	....
..	$t_0 = 56.9°$	$x = 831$	....	....

$p(t - t_0)^2$	$p(t - t_0)(l - x)$	$y(t - t_0)$	v	pvv
5184.20	6440.0	− 40.6	− 0.6	1.8
392.04	396.0	− 24.9	− 4.9	24.0
138.24	57.6	+ 6.0	+ 4.0	96.0
346.56	501.6	− 9.6	+ 1.4	11.8
784.08	1267.2	+ 12.5	− 3.5	98.0
1705.28	2102.4	+ 18.4	+ 0.4	1.3
8550.40	10,764.8	....	....	232.9
$y = 1.26$		....	....	....

$$\mu = \sqrt{\frac{232.9}{6-2}} = 7.6,$$

$$\mu_x = \frac{7.6}{\sqrt{34}} = 1.3,$$

$$\mu_y = \frac{7.6}{\sqrt{8550.40}} = 0.1.$$

Value of $x = 831$ at $56.9°$

$$\begin{array}{rr} 6.4 & 5.1° \\ \hline = 837.4 & 62.0° \end{array}$$

$$\mu_{62}^2 = (1.3)^2 + (5.1)^2 \times (0.1)^2 = 1.9.$$

These values may be checked by computing by the ordinary process.

CHAPTER V

ADJUSTMENT OF CONDITIONED OBSERVATIONS

117. We now take up the third division of the subject as laid down in Art. 25. So far the quantities we have dealt with, whether directly observed or functions of the quantities observed, have been independent of one another; but if they are not independent of one another — that is, if they must satisfy exactly certain relations that exist *à priori* and are entirely separate from any relations demanded by observation — they are said to be *conditioned* by these relations, or the relations are spoken of as conditions.

All problems relating to conditioned observations may be solved by the rules laid down in the preceding chapters.

Let, with the usual notation, $X_1, X_2, \ldots X_n$ denote the most probable values of n directly observed quantities $M_1, M_2, \ldots M_n$ whose weights are $p_1, p_2, \ldots p_n$ respectively. Let the n_c conditions to be satisfied exactly by the most probable values, when expressed by equations reduced to the linear form, be

$$\begin{aligned} a'X_1 + a''X_2 + \cdots - L' &= 0, \\ b'X_1 + b''X_2 + \cdots - L'' &= 0, \\ \cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{1}$$

where $a', a'', \ldots; b', b'', \ldots; \ldots; L', L'', \ldots$ are known constants.

If $v_1, v_2, \ldots v_n$ denote the most probable corrections to the observed values, so that

$$\begin{aligned} X_1 - M_1 &= v_1, \\ X_2 - M_2 &= v_2, \\ \cdots\cdots \\ X_n - M_n &= v_n, \end{aligned} \tag{2}$$

we have the reduced condition equations

$$
\begin{aligned}
a'v_1 + a''v_2 + \cdot \cdot \cdot - l' &= 0,\\
b'v_1 + b''v^2 + \cdot \cdot \cdot - l'' &= 0,\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot
\end{aligned}
$$

or

$$
\begin{aligned}
[av] - l' &= 0,\\
[bv] - l'' &= 0, \qquad (3)\\
\cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot
\end{aligned}
$$

where $l' = L' - [aM]$, $l'' = L'' - [bM]$, . . . , and are, therefore, known quantities.

The most probable system of corrections is that which makes

$$[pv^2] = \text{a minimum, } \omega \text{ suppose.}$$

The problem is to solve this minimum function when the corrections v are subject to the above n_c conditions.

Direct Solution — Method of Independent Unknowns.

118. It is plain that n_c of the corrections can, by means of the condition equations, be expressed in terms of the remaining $n - n_c$ corrections, and that by substituting these n_c values in the minimum function, we should have a reduced minimum function containing $n - n_c$ independent unknowns. This function can be found in the usual way by equating to zero its differential coefficients with respect to each unknown in succession. The $n - n_c$ resulting equations, taken in connection with the n_c condition equations, determine the n corrections v_1, v_2, . . . v_n. Thence $[pvv]$ is found.

The solution of the $n - n_c$ equations can be carried through by any of the methods of Chapter IV. The precision of the adjusted values, or of any function of them, can also be found as in Chapter IV.

Ex. 1. — Take that already solved in Ex. 4, Art. 77.

Let v_1, v_2, v_3, v_4, v_5 be the most probable corrections to the measured angles, then the conditions to be satisfied are

$$
\begin{aligned}
PSB + v_4 &= FSB + v_3 - FSP - v_1,\\
OSB + v_5 &= FSB + v_3 - FSO - v_2.
\end{aligned}
$$

Substituting for PSB, FSB, etc., their measured values, the condition equations may be written

$$v_1 - v_3 + v_4 = -0.76,$$
$$v_2 - v_3 + v_5 = -1.66,$$

with
$$5v_1^2 + 7v_2^2 + 4v_3^2 + 7v_4^2 + 4v_5^2 = \text{a min.}$$

Substitute for v_4, v_5 in the minimum equation, and

$$5v_1^2 + 7v_2^2 + 4v_3^2 + 7(v_1 - v_3 + 0.76)^2 + 4(v_2 - v_3 + 1.66)^2 = \text{a min.}$$

Hence, differentiating with respect to v_1, v_2, v_3, as independent variables, we have the normal equations

$$\begin{aligned} 12v_1 \qquad\quad - 7v_3 &= -5.32, \\ 11v_2 - 4v_3 &= -6.64, \\ -7v_1 - 4v_2 + 15v_3 &= 11.96, \end{aligned}$$

whence
$$v_1 = -0.05'', \qquad v_2 = -0.36'', \qquad v_3 = +0.68'';$$

and from the condition equations,

$$v_4 = -0.03'', \qquad v_5 = -0.62''.$$

These results are the same as those already found in Art. 77.

Ex. 2. — The angles A, B, C, of a spherical triangle are equally well measured; required the adjusted values and their weights.

The condition equation to be satisfied is

$$A + B + C = 180 + \epsilon, \tag{1}$$

where ϵ is the spherical excess of the triangle.

Putting $M_1 + v_1$, $M_2 + v_2$, $M_3 + v_3$ for A, B, C, the condition equation becomes

$$\begin{aligned} v_1 + v_2 + v_3 &= 180 + \epsilon - [M] \\ &= l, \text{ suppose.} \end{aligned} \tag{2}$$

Also
$$v_1^2 + v_2^2 + v_3^2 = \text{a min.}$$

Substitute for v_3 from (2) in the minimum function, and

$$v_1^2 + v_2^2 + (v_1 + v_2 - l)^2 = \text{a min.}$$

Differentiating with respect to the independent variables v_1, v_2, and

$$\begin{aligned} 2v_1 + v_2 &= l, \\ v_1 + 2v_2 &= l, \end{aligned} \tag{3}$$

which give
$$v_1 = v_2 = \frac{l}{3}.$$

Also, from Eq. 2,
$$v_3 = \frac{l}{3}.$$

Hence *the correction to each angle is one-third of the difference of the theoretical and measured sums of the three angles.*

To find the weight of the adjusted value of an angle, as A.

The function is $$dF = v_1.$$
Hence, following the method of Art. 101 (*b*),

$$u_F = \frac{1}{\text{wt.}} = [GQ],$$

where $G_1 = 1$, and Q_1, Q_2 are found from

$$2\,Q_1 + \quad Q_2 = 1,$$
$$Q_1 + 2\,Q_2 = 0;$$

that is, weight of $A = \frac{3}{2}$ if weight of measured value is unity.

CHECK. Weight of direct measure of A . . . . $= 1$,
Wt. of indirect meas. ($= 180 + \epsilon - B - C$) of $A = \frac{1}{2}$,
Weight of mean . . . . $= 1\frac{1}{2}$,

as already found.

Ex. 3. — To find the weight of a side, a, in a triangle, all of whose angles have been equally well measured, the base, b, being free from error.

Here $$F = a = b\,\frac{\sin A}{\sin B},$$

$$\therefore\ dF = a \sin 1'' (\cot A v_1 - \cot B v_2).$$

The weight is found from

$$u_F = a \sin 1'' \cot A Q_1 - a \sin 1'' \cot B Q_2,$$

where Q_1, Q_2 satisfy the equations (Art. 101),

$$2\,Q_1 + \quad Q_2 = \quad a \sin 1'' \cot A,$$
$$Q_1 + 2\,Q_2 = - a \sin 1'' \cot B.$$

Hence, $$u_F = \tfrac{2}{3}\,a^2 \sin^2 1'' (\cot^2 A + \cot^2 B + \cot A \cot B).$$

Indirect Solution — Method of Correlates.

119. If the unknowns in the condition equations are much entangled the direct solution would be very laborious. It is in general, therefore, advisable, instead of eliminating the n_c unknowns directly, to do so indirectly by means of undetermined multipliers, or correlates, as they are called.

If we multiply the condition equations (3), Art. 117, in order by the correlates C_1, C_2, . . ., we may write

$$\omega = C_1\,([av] - l') + C_2\,([bv] - l'') + \cdots + [pvv] = \text{a min.} \qquad (1)$$

and determine C_1, C_2, . . ., accordingly.

By differentiation,

$$d\omega = (a'C_1 + b'C_2 + \cdots + 2p_1v_1)\,dv_1$$
$$+ (a''C_1 + b''C_2 + \cdots + 2p_2v_2)\,dv_2 + \cdots \qquad (2)$$

If we place equal to zero the coefficients of n_c of the differentials dv_1, dv_2, . . ., we shall have n_c equations from which to find C_1, C_2, . . . Substitute these n_c values in the expression for $d\omega$, and there will remain $n - n_c$ differentials which are independent of one another. In order that the function may satisfy the condition of a minimum, the coefficients of each of these differentials must be equal to zero. This gives $n - n_c$ equations, which equations, taken in connection with the n_c condition equations, give the n unknowns v_1, v_2, . . . v_n.

The practical solution would, therefore, be: Form n equations by placing equal to zero the differential coefficients of the minimum function with respect to each of the quantities v_1, v_2, . . . v_n. From these n equations and the n_c condition equations determine the $n + n_c$ unknowns C_1, C_2, . . ., v_1, v_2, . . ., and thence the function $[pvv]$.

In carrying this out the form of the differential equation (2) shows that it would be advantageous to multiply the minimum equation by $-\frac{1}{2}$, and so write (1) in the form

$$C_1([av] - l') + C_2([bv] - l'') + \cdots - \tfrac{1}{2}[pvv] = \text{a min.} \qquad (3)$$

Differentiating, we have the n correlate equations

$$\begin{aligned} a'C_1 + b'C_2 + \cdots &= p_1v_1, \\ a''C_1 + b''C_2 + \cdots &= p_2v_2, \\ \cdots\cdots&\cdots\cdots \end{aligned} \qquad (4)$$

Substituting for v_1, v_2, . . . in the condition equations their values derived from these equations, and the normal equations result. They are

$$\begin{aligned} \left[\frac{aa}{p}\right]C_1 + \left[\frac{ab}{p}\right]C_2 + \cdots &= l', \\ \left[\frac{ab}{p}\right]C_1 + \left[\frac{bb}{p}\right]C_2 + \cdots &= l'', \\ \cdots\cdots&\cdots\cdots \end{aligned} \qquad (5)$$

Solving, we obtain $C_1, C_2, \ldots$, and thence $v_1, v_2, \ldots$ from (4), and $X_1, X_2, \ldots$ from (2), Art. 117.

The normal equations may be written

$$
\begin{aligned}
[uaa]\, C_1 + [uab]\, C_2 + \cdots &= l', \\
[uab]\, C_1 + [ubb]\, C_2 + \cdots &= l'', \qquad (6) \\
\cdots\cdots\cdots\cdots
\end{aligned}
$$

where $u_1, u_2, \ldots$ denote the reciprocals of the weights $p_1, p_2, \ldots$ The form of these equations shows that the coefficients $[uaa]$, $[uab]$, ... may be computed as in Art. 79, the corresponding scheme being,

		C_1	C_2	C_3	...
v_1	u_1	a'	b'	c'	...
v_2	u_2	a''	b''	c''	...
.	.	.	.	.	...

If the elimination of the normal equations is performed by the method of substitution (Art. 84), we have, by collecting the first equations of the successive groups,

$$
\begin{aligned}
[uaa]\, C_1 + [uab]\ \ C_2 + [uac]\ \ C_3 + \cdots &= l' \\
+ [ubb.1]\, C_2 + [ubc.1]\, C_3 + \cdots &= l''.1 \\
+ [ucc.2]\, C_3 + \cdots &= l'''.2 \qquad (7) \\
\cdots\cdots\cdots\cdots
\end{aligned}
$$

where l', $l''.1$, $l'''.2$, correspond to $[al]$, $[bl.1]$, $[cl.2]$, ... respectively.

These equations being precisely similar in form to Eq. 8, Art. 84, the elimination gives (see Art. 101),

$$
\begin{aligned}
C_1 &= \frac{l'}{[uaa]} + \frac{l''.1}{[ubb.1]} R' + \frac{l'''.2}{[ucc.2]} R'' + \cdots \\
C_2 &= \qquad\quad \frac{l''.1}{[ubb.1]} \quad + \frac{l'''.2}{[ucc.2]} S'' + \cdots \qquad (8) \\
\cdots\cdots\cdots\cdots
\end{aligned}
$$

where

$$
0 = \frac{[uab]}{[uaa]} + R',
$$

$$0 = \frac{[uac]}{[uaa]} + \frac{[ubc.1]}{[ubb.1]} R' + R'', \tag{9}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$0 = \frac{[ubc.1]}{[ubb.1]} + S'',$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\begin{aligned} l''.1 &= l'R' + l'', \\ l'''.2 &= l'R'' + l''S'' + l''', \end{aligned} \tag{10}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

120. **Ex. 1.**—Take that solved in Ex. 1, Art. 118. The condition equations are

$$\begin{aligned} v_1 - v_3 + v_4 &= -0.76, \\ v_2 - v_3 + v_5 &= -1.66. \end{aligned} \tag{1}$$

The correlate equations consequently are

$$\begin{aligned} C_1 &= 5\,v_1, \\ C_2 &= 7\,v_2, \\ -C_1 - C_2 &= 4\,v_3, \\ C_1 &= 7\,v_4, \\ C_2 &= 4\,v_5. \end{aligned} \tag{2}$$

To form the normal equations, we may substitute for $v_1, v_2, \ldots$ from (2) in (1), or proceed by means of the tabular form in Art. 119. We find

$$\begin{aligned} 0.59\,C_1 + 0.25\,C_2 &= -0.76, \\ 0.25\,C_1 + 0.64\,C_2 &= -1.66. \end{aligned}$$

The solution of these equations gives

$$C_1 = -0.23, \qquad C_2 = -2.49,$$

whence, from the correlate equations,

$$v_1 = -0.05'', \quad v_2 = -0.36'', \quad v_3 = +0.68'', \quad v_4 = -0.03'', \quad v_5 = -0.62''.$$

CHECK.—The results satisfy the condition equations.

Ex. 2.—The angles A, B, C, of a spherical triangle are measured with their weights, p_1, p_2, p_3; required their adjusted values.

The condition equation may be written (see Ex. 2, Art. 118)

$$v_1 + v_2 + v_3 = l,$$

with $\qquad [pv^2] = \text{a min.}$

The correlate equations are

$$\begin{aligned} C &= p_1 v_1, \\ C &= p_2 v_2, \\ C &= p_3 v_3, \end{aligned}$$

and the normal equation $\qquad [u]\,C = l.$

$$\therefore\; v_1 = \frac{u_1}{[u]}\,l, \qquad v_2 = \frac{u_2}{[u]}\,l, \qquad v_3 = \frac{u_3}{[u]}\,l.$$

Hence the adjusted values are known.

When the weights are equal, then

$$v_1 = v_2 = v_3 = \tfrac{1}{3}\,l,$$

the same results as in Ex. 2, Art. 118.

NOTE. — If a condition equation is of the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = l,$$

the weights of the measured values being $p_1, p_2 \ldots p_n$, then, proceeding as in the above, we have

$$v_1 = \frac{u_1 a_1}{[uaa]} l, \quad v_2 = \frac{u_2 a_2}{[uaa]} l, \ldots$$

This result is very important, and will be often referred to.

Ex. 3. — At U.S. Coast Survey station, Pine Mountain, the following were the angles observed between the surrounding stations in order of azimuth:

Jocelyne-Deepwater .	65°	11′	52.500″,	weight	3,
Deepwater-Deakyne .	66°	24′	15.553″,	"	3,
Deakyne-Burden . .	87°	02′	24.703″,	"	3,
Burden-Jocelyne . .	141°	21′	21.757″,	"	1,

required their most probable values.

The condition to be satisfied is that the sum of the angles should be 360°. Now,

Sum of measured values	=	359°	59′	54.513″
Theoretical sum . . .	=	360°	00′	00.000″
∴ Residual error . . .	=			5.487″

Hence, as in the preceding example,

$$\text{Correction to each of first three angles} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 1} \times 5.487'' = 0.914''.$$

$$\text{Correction to fourth angle} \quad \ldots \ldots = \frac{1}{\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 1} \times 5.487'' = 2.744''.$$

Ex. 4. — "In order to find the content of a piece of ground, I measured with a common circumferentor and chain the bearings and lengths of its several sides. But upon casting up the difference of latitude and departure, I discovered that some error had been contracted in taking the dimensions. Now, it is required to compute the area of this inclosure on the *most probable supposition* of this error.

"Let $ABCDE$ be a survey accurately protracted according to the measured lengths and bearings of the sides AB, BC, . . . A the place of beginning, E of ending, AG a meridian, AF, FE the errors in latitude and departure. Now, the problem requires us to make such changes in the positions of the points B, C, . . . that we may remove the errors AF, FE — in other words, that E may coincide with A; and these changes must be made in the most probable manner. We have, therefore, to fulfill the three following conditions:

"All the changes in departure must remove the error in departure EF.

"All the changes in latitude must remove the error in latitude AF.

"The probability of these changes must be a maximum."

Let $a_1, a_2, a_3, \ldots; \theta_1, \theta_2, \theta_3, \ldots$; denote the measured lengths and bearings of the sides AB, BC, . . . , and $x_1, x_2, x_3, \ldots; y_1, y_2, y_3, \ldots$ their most probable corrections.

Now, since the corrected latitudes must balance, and the corrected departures must also balance, we have the conditions

$$(a_1 + x_1) \cos (\theta_1 + y_1) + (a_2 + x_2) \cos (\theta_2 + y_2) + \cdots = 0,$$
$$(a_1 + x_1) \sin (\theta_1 + y_1) + (a_2 + x_2) \sin (\theta_2 + y_2) + \cdots = 0,$$

or, reducing to the linear form,

$$\begin{aligned} &\cos \theta_1 x_1 - a_1 \sin \theta_1 y_1 + \cos \theta_2 x_2 - a_2 \sin \theta_2 y_2 + \cdots + [a \cos \theta] = 0, \\ &\sin \theta_1 x_1 + a_1 \cos \theta_1 y_1 + \sin \theta_2 x_2 + a_2 \cos \theta_2 y_2 + \cdots + [a \sin \theta] = 0, \end{aligned} \qquad (1)$$

with

$$[px^2] + [qy^2] = \text{a minimum},$$

the weights of $x_1, x_2, \ldots; y_1, y_2, \ldots$ being $p_1, p_2, \ldots; q_1, q_2, \ldots$ respectively.

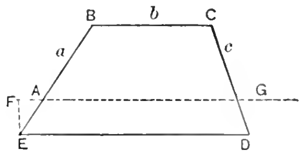

Fig. 4.

Hence the correlate equations,

$$\begin{aligned} + \cos \theta_1 C_1 + \sin \theta_1 C_2 &= p_1, x_1, \\ - a_1 \sin \theta_1 C_1 + a_1 \cos \theta_1 C_2 &= q_1, y_1, \end{aligned} \qquad (2)$$

and the normal equations,

$$\left\{\left[\frac{\cos^2 \theta}{p}\right] + \left[\frac{a^2 \sin^2 \theta}{q}\right]\right\} C_1 + \left\{\left[\frac{\sin \theta \cos \theta}{p}\right] - \left[\frac{a^2 \sin \theta \cos \theta}{q}\right]\right\} C_2 = -[a \cos \theta],$$

$$\left\{\left[\frac{\sin \theta \cos \theta}{p}\right] - \left[\frac{a^2 \sin \theta \cos \theta}{q}\right]\right\} C_1 + \left\{\left[\frac{\sin^2 \theta}{p}\right] + \left[\frac{a^2 \cos^2 \theta}{q}\right]\right\} C_2 = -[a \sin \theta].$$

Now if we assume

$$p_1 = \frac{1}{a_1}, \quad p_2 = \frac{1}{a_2}, \ldots$$
$$q_1 = a_1, \quad q_2 = a_2, \ldots$$

as "this seems best to agree with the imperfections of the common instruments used in surveying," the normal equations reduce to

$$\begin{aligned} C_1 [a] &= -[a \cos \theta], \\ C_2 [a] &= -[a \sin \theta], \end{aligned}$$

from which C_1, C_2 are known.

The weights indicated correspond to the assumption that the errors of measurement of the lengths are directly, and of the same bearings inversely, proportional to the square root of the lengths of the corresponding courses.

The corrections $x_1, x_2, \ldots; y_1, y_2 \ldots$, are known from (2).

The errors in latitude (see Eq. 1) now reduce to

$$\cos \theta_1 x_1 - a_1 \sin \theta_1 y_1 = -a_1 \frac{[a \cos \theta]}{[a]},$$

$$\cos \theta_2 x_2 - a_2 \sin \theta_2 y_2 = - a_2 \frac{[a \cos \theta]}{[a]},$$

$$\cdots\cdots\cdots\cdots\cdots$$

and the errors in departure to

$$\sin \theta_1 x_1 + a_1 \cos \theta_1 y_1 = - a_1 \frac{[a \sin \theta]}{[a]},$$

$$\sin \theta_2 x_2 + a_2 \sin \theta_2 y_2 = - a_2 \frac{[a \sin \theta]}{[a]},$$

$$\cdots\cdots\cdots\cdots\cdots$$

Hence Bowditch's *rule for balancing a survey*: "Say as the sum of all the distances is to each particular distance, so is the whole error in departure to the correction of the corresponding departure, each correction being so applied as to diminish the whole error in departure. Proceed in same way for the correction in latitude."

This problem was proposed as a prize question by Robert Patterson, of Philadelphia, in vol. i. No. 3 of the *Analyst or Mathematical Museum*, edited by Dr. Adrain, of Reading, Pa., and published in 1807. In vol. i. No. 4 are two solutions — one by Bowditch, to whom the prize was awarded, and the other by Dr. Adrain. Adrain's mode of solution is nearly the same as by the ordinary Gaussian method. He employs undetermined multipliers or correlates, exactly as Gauss subsequently did. To Adrain, therefore, is due not only the first derivation of the exponential law of error, but its first application to geodetic work.

To Find the Precision of the Adjusted Values, or of any Function of them.

121. The method of proceeding is the same as in Art. 108.

The first step is to find r, the p. e. of a single observation, and next the weight, p_F, of the function, whence the p. e. of the function is given by

$$r \sqrt{u_F},$$

u_F being the reciprocal of the weight.

(*a*) To find r.

In Art. 105 it was shown that in a system of observation equations the p. e. r of an observation of the unit of weight is found from

$$r = 0.6745 \sqrt{\frac{[pv^2]}{n - n_i}}$$

where $[pv^2]$ is the sum of the weighted squares of the residuals, v, n is the number of observation equations and n_i the number of independent unknowns.

Hence, in a system of condition equations, n being the number of observed quantities and n_c the number of conditions, the number of independent unknowns is $n - n_c$, and

$$r = 0.6745 \sqrt{\frac{[pv^2]}{n - (n - n_c)}}$$
$$= 0.6745 \sqrt{\frac{[pv^2]}{n_c}}. \qquad (1)$$

Lüroth's formula (Art. 105) may be used as a check on the value of r.

Checks of $[pv^2]$. — When the number of residuals is large, in order to guard against mistakes $[pv^2]$ should be computed in at least two different ways. The following check methods will be found useful:

(α) The correlate equations 4, Art. 119, may be written,

$$\sqrt{p_1}v_1 = \sqrt{u_1}a'C_1 + \sqrt{u_1}b'C_2 + \cdots$$
$$\sqrt{p_2}v_2 = \sqrt{u_2}a''C_1 + \sqrt{u_2}b''C_2 + \cdots$$
$$\cdots\cdots\cdots\cdots$$

Square and add, and

$$\begin{aligned}[pv^2] = [uaa]\,C_1C_1 &+ 2\,[uab]\,C_1C_2 + 2\,[uac]\,C_1C_3 + \cdots \\ &+ \quad [ubb]\,C_2C_2 + 2\,[ubc]\,C_2C_3 + \cdots \\ &\qquad\qquad + \quad [ucc]\,C_3C_3 + \cdots \qquad (2) \\ &\qquad\qquad\qquad\qquad + \cdots\end{aligned}$$

$= [Cl]$ from (6), Art. 119.

(β) $[pv^2] = [Cl]$

$$\begin{aligned} &= l'C_1 + l''C_2 + \cdots \\ &= l'\left(\frac{l'}{[uaa]} + \frac{l''.1}{[ubb.1]}R' + \frac{l'''.2}{[ucc.2]}R'' + \cdots\right) \\ &\qquad + l''\left(\frac{l''.1}{[ubb.1]} + \frac{l'''.2}{[ucc.2]}S'' + \cdots\right) \\ &\qquad\qquad + \cdots\cdots\cdots \\ &= \frac{(l')^2}{[uaa]} + \frac{(l''.1)^2}{[ubb.1]} + \frac{(l'''.2)^2}{[ucc.2]} + \cdots \qquad (3)\end{aligned}$$

by addition attending to Eq. 10, Art. 119.

This expression is very readily computed from the solution of the correlate normal equations, as shown in Ex. 2 following. Compare the computation of $[vv]$ from the scheme in Art. 106.

The sum $[pv^2]$ can in general be computed more rapidly by these methods than by the direct process of summing the weighted squares of the residuals.

122. Ex. 1. — The three angles of a triangle are measured with the weights p_1, p_2, p_3; required the mean-square error of a single observation.

Using the values of v_1, v_2, v_3, found in Ex. 2, Art. 120, we have

$$[pv^2] = \frac{u_1 l^2}{[u]^2} + \frac{u_2 l^2}{[u]^2} + \frac{u_3 l^2}{[u]^2}$$
$$= \frac{l^2}{[u]}.$$

Hence
$$\mu = \frac{l}{\sqrt{[u]}}, \quad \text{since } n_c = 1.$$

CHECK (1).
$$[pv^2] = [Cl]$$
$$= \frac{l}{[u]} l,$$

as before.

CHECK (2). $[pv^2] = \dfrac{(l)^2}{[u]}$ directly from Eq. 3, since $[uaa] = 1$.

Ex. 2. — To find the p. e. of a single observation in Ex. 1, Art. 120.

The first step is to find the value of $[pv^2]$. Three methods are given:

(1)

p	v	pv^2
5	− 0.05	.01
7	− 0.36	.91
4	+ 0.68	1.85
7	− 0.03	.01
4	− 0.62	1.54
. .	. . . .	4.32 = $[pv^2]$

(2)

C	l	Cl
− 0.230	− 0.76	0.17
− 2.493	− 1.66	4.14
. . . .	. . . .	4.31 = $[pv^2]$

(3) From the solution of the correlate normal equations:

C_1	C_2	
$+0.5929$	$+0.2500$	$-0.76 = l'$
$+0.2500$	$+0.6429$	$-1.66 = l''$
$+1$	$+0.4217$	$-1.2818 = \frac{l'}{[u.aa]}$
. . . .	$+0.5375$	$-1.3395 = l''.1$
. . . .	$+1$	$-2.492 = \frac{l''.1}{[ubb.1]}$

$$\therefore [pv^2] = 0.76 \times 1.2819 + 1.3395 \times 2.492$$
$$= 4.3136.$$

Hence, the number of conditions being two,

$$r = 0.6745\sqrt{\frac{4.31}{2}} = 0.99''.$$

123. (*b*) To find u_F.

Let the function whose weight is to be found be

$$F = f(V_1, V_2, \dots V_n), \tag{4}$$

and let it be conditioned by the n_c equations

$$\begin{aligned} f_1(V_1, V_2, \dots V_n) &= 0, \\ f_2(V_1, V_2, \dots V_n) &= 0, \\ \dots\dots\dots\dots \end{aligned} \tag{5}$$

Expressing F in terms of the observed values, $M_1, M_2, \dots M_n$, which are independent of one another, and reducing to the linear form, we have

$$dF = \frac{\delta F}{\delta M_1} v_1 + \frac{\delta F}{\delta M_2} v_2 + \cdots \tag{6}$$

Hence, as in Art. 108,

$$u_F = u_1\left(\frac{\delta F}{\delta M_1}\right)^2 + u_2\left(\frac{\delta F}{\delta M_2}\right)^2 + \dots \tag{7}$$

where $u_1, u_2, \dots$ are the reciprocals of the weights of the observed values.

Ex. 3. — To find the m. s. e. of a side, a, in a triangle whose angles have been measured with the weights p_1, p_2, p_3, the base, b, being free from error.

The function equation is

$$F = a = b\frac{\sin A}{\sin B},$$

and the condition equation

$$A + B + C = 180 + \epsilon.$$

Hence from Ex. 2, Art. 120, expressing A, B in terms of the observed values,

$$A = M_1 + \frac{u_1}{[u]}\{180 + \epsilon - (M_1 + M_2 + M_3)\},$$

$$B = M_2 + \frac{u_2}{[u]}\{180 + \epsilon - (M_1 + M_2 + M_3)\}.$$

Now,

$$dF = \left(\frac{\delta F}{\delta A}\frac{\delta A}{\delta M_1} + \frac{\delta F}{\delta B}\frac{\delta B}{\delta M_1}\right)v_1$$
$$+ \left(\frac{\delta F}{\delta A}\frac{\delta A}{\delta M_2} + \frac{\delta F}{\delta B}\frac{\delta B}{\delta M_2}\right)v_2 + \left(\frac{\delta F}{\delta A}\frac{\delta A}{\delta M_3} + \frac{\delta F}{\delta B}\frac{\delta B}{\delta M_3}\right)v_3$$
$$= a \sin 1''\left(\left\{\left(1 - \frac{u_1}{[u]}\right)\cot A + \frac{u_2}{[u]}\cot B\right\}v_1\right.$$
$$\left. + \left\{-\frac{u_1}{[u]}\cot A - \left(1 - \frac{u_2}{[u]}\right)\cot B\right\}v_2 + \left\{-\frac{u_1}{[u]}\cot A + \frac{u_2}{[u]}\cot B\right\}v_3\right).$$

Therefore,

$$u_F = a^2 \sin^2 1''\left(\left\{\left(1 - \frac{u_1}{[u]}\right)\cot A + \frac{u_2}{[u]}\cot B\right\}^2 u_1\right.$$
$$\left. + \left\{\frac{u_1}{[u]}\cot A + \left(1 - \frac{u_2}{[u]}\right)\cot B\right\}^2 u_2 + \left\{\frac{u_1}{[u]}\cot A - \frac{u_2}{[u]}\cot B\right\}^2 u_3\right)$$
$$= a^2 \sin^2 1''\left\{\left(u_1 - \frac{u_1^2}{[u]}\right)\cot^2 A + \left(u_2 - \frac{u_2^2}{[u]}\right)\cot^2 B + \frac{2\,u_1 u_2}{[u]}\cot A \cot B\right\}$$

and

$$\mu_F = \mu\sqrt{u_F},$$

where μ is the m. s. e. of a single observation.

If the weights p_1, p_2, p_3 are each equal to unity, this reduces to

$$u_F^2 = \tfrac{2}{3}\, a^2 \sin^2 1''\, \mu^2 (\cot^2 A + \cot^2 B + \cot A \cot B),$$

and if the triangle is equilateral,

$$\mu_F^2 = \tfrac{2}{3}\, a^2 \sin^2 1''\, \mu^2.$$

Also, if the base, instead of being considered exact, had the m. s. e. μ_b the expressions for μ_F^2 would be increased by $\frac{a^2}{b^2}\mu_b^2$ and μ_b^2 respectively.

124. It is usually more convenient in practice to use the method of correlates.

Let the function, reduced to the linear form, be written

$$dF = f'v_1 + f''v_2 + \cdots \tag{1}$$

This is conditioned by the n_c equations, also in the linear form,

$$a'v_1 + a''v_2 + \cdot \cdot \cdot - l' = 0,$$
$$b'v_1 + b''v^2 + \cdot \cdot \cdot - l'' = 0, \tag{2}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

with

$$[pv^2] = \text{a minimum}.$$

Referring to the principle of Art. 119, we see that by using correlates C_1, C_2, . . ., and determining them properly, we can express the function in terms of the quantities v'_1, v'_2, . . . v'_n as if independent; that is,

$$dF = (f' - a'C_1 - b'C_2 - \cdot \cdot \cdot)v'_1$$
$$+ (f'' - a''C_1 - b''C_2 - \cdot \cdot \cdot)v'_2 + \cdot \cdot \cdot \tag{3}$$

and, therefore,

$$u_F = (f' - a'C_1 - b'C_2 - \cdot \cdot \cdot)^2 u_1$$
$$+ (f'' - a''C_1 - b''C_2 - \cdot \cdot \cdot)^2 u_2 + \cdot \cdot \cdot \tag{4}$$

It remains to determine C_1, C_2, . . . Now, when the most probable values of the corrections v'_1, v'_2, . . . v'_n are substituted in the value of the function dF, this function must have its most probable value, and, therefore, its maximum weight. We may, therefore, determine the correlates C from the condition that the weight of dF is a maximum; that is, that u_F is a minimum. Differentiate, then, u_F with respect to C_1, C_2 . . . as independent variables, and we have the equations,

$$[uaa]\,C_1 + [uab]\,C_2 + \cdot \cdot \cdot = [uaf],$$
$$[uab]\,C_1 + [ubb]\,C_2 + \cdot \cdot \cdot = [ubf], \tag{5}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

from which C_1, C_2 . . . are found.

These equations being precisely of the form of ordinary normal equations, it follows, as in (*c*) and (*d*), Art 106, that

$$u_F = [uff] - [uaf]\,C_1 - [ubf]\,C_2 - \cdot \cdot \cdot \tag{6}$$

or

$$u_F = [uff] - \frac{[uaf]^2}{[uaa]} - \frac{[ubf.1]^2}{[ubb.1]} - \cdot \cdot \cdot \tag{7}$$

The form of the last expression for u_F shows that it may be found by means of the following scheme, in which $[uaf]$, $[ubf]$, . . . are added as an extra column in the solution of the corre-

late normal equations (5), in the manner shown in Art. 106. For three correlates the scheme would be

C_1	C_2	C_3	
$[uaa]$	$[uab]$ $[ubb]$	$[uac]$ $[ubc]$ $[ucc]$	$[uaf]$ $[ubf]$ $[ucf]$ $[uff]$
. . . .	$[ubb.1]$	$[ubc.1]$ $[ucc.1]$	$[ubf.1]$ $[ucf.1]$ $[uff.1]$
. . . .	. . . .	$[ucc.2]$	$[ucf.2]$ $[uff.2]$
. . . .	. . . .	. . . .	$[uff3]$ $= u_F$

125. Ex. 4. — To find the weight of the angle PSB in Ex. 1, Art. 118. Here

$$dF = - v_1 + v_3,$$

$$\therefore f_1 = -1, \qquad f_2 = 0, \qquad f_3 = +1.$$

From the condition equations

$$a' = +1, \qquad b' = +1,$$
$$a''' = -1, \qquad b''' = -1,$$
$$a''''' = +1, \qquad b''''' = +1,$$
$$\therefore [uaf] = \tfrac{1}{5} \times -1 + \tfrac{1}{4} \times -1 = -0.45,$$
$$[ubf] = -\tfrac{1}{4},$$
$$[ucf] = +0.45.$$

The correlate normal equations with the extra column for finding u_F:

C_1	C_2	l		
+ 0.5929 + 0.2500	+ 0.2500 + 0.6429	− 0.7600 − 1.6600	− 0.4500 = $[uaf]$ + 0.2500 = $[ubf]$	+ 0.4500 = $[uff]$
+ 1	+ 0.4217 + 0.5375	− 1.2818 − 1.3395	− 0.7590 − 0.0602 = $[ubf.1]$	+ 0.3416 + 0.1084 = $[uff.1]$
. . . .	+ 1	− 2.492	− 0.1120	+ 0.0067 + 0.1017 = $[uff.2]$ $= u_F$

Also
$$\mu_F = \mu \sqrt{u_F}$$
$$= 1.47 \sqrt{0.1017} \text{ from Ex. 2}$$
$$= 0.47'', \text{ as before.}$$

Ex. 5. — To find the weight and m. s. e. of the adjusted value of an angle of a triangle when all three angles are measured, their weights being p_1, p_2, p_3, respectively.

The function is $dF = v'_1$,

and the condition equation $v_1 + v_2 + v'_3 = l$.

Hence from (15)
$$u_F = u_1 - \frac{u_1^2}{[u]}$$
$$= \frac{u_1(u_2 + u_3)}{[u]}.$$

Also
$$\mu_F = \mu \sqrt{u_F}$$
$$= \frac{l}{\sqrt{[u]}} \sqrt{\frac{u_1(u_2+u_3)}{[u]}} \text{ (See Ex. 1)}$$
$$= \frac{l}{[u]} \sqrt{u_1(u_2 + u_3)}.$$

The weight of an angle before adjustment is to the weight after adjustment as
$$\frac{1}{u_1} : \frac{[u]}{u_1(u_2+u_3)},$$
or
$$u_2 + u_3 : [u].$$

If $p_1 = p_2 = p_3 = 1$, the weights are as 2 : 3. This result is independent of the magnitude of the angle. It therefore applies to any problem in which the condition to be satisfied is that the sum of two quantities shall be equal to a third, or in which the sum of all three is equal to a constant. For other solutions, see Ex. 2, Art. 118.

Ex. 6. — If n angles measured at a station close the horizon, find the weight of the adjusted value of any one of them.

[The solution is exactly as in the preceding example.

The weight of V'_1, for instance, is found from
$$u_{V'_1} = \frac{u_1\{[u] - u_1\}}{[u]}.$$

If the weights $p_1, p_2, \ldots$ are all equal to one another, the weight of an angle after adjustment is to its weight before adjustment as
$$n : n - 1].$$

Ex. 7. — Show that the weight of the sum of the adjusted angles of a triangle is infinite.

[Sum $= 180 + \epsilon$, a fixed quantity,
$$\therefore \text{p. e.} = 0, \text{ and weight} = \infty,$$
or otherwise
$$v'_1 + v'_2 + v'_3 = l,$$
$$a' = a'' = a''' = 1,$$

$$\therefore u_F = 3 - \frac{3^2}{3} = 0].$$

Ex. 8. — In the "longitude triangle" Brest, Greenwich, Paris, as determined by the U. S. Coast Survey in 1872, the observed values were

	m.	s.		
Brest-Greenwich .	17	57.154,	weight	10,
Greenwich-Paris .	9	21.120,	"	7,
Brest-Paris . . .	27	18.190,	"	9.

Show that the most probable values are

m.	s.		
17	57.130,	weight	14
9	21.086,	"	12
27	18.216,	"	13

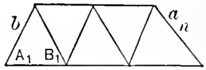

Fig. 5.

126. Ex. 9. — To find the weight of a side in a chain of triangles, all of the angles of each triangle having been equally well measured and the base being free from error.

Let b be the measured value of the base, and let $a_1, a_2, \ldots a_n$ be the sides of continuation in order as computed from b; a_n being the side whose weight is required.

If $A_1, B_1, A_2, B_2, \ldots$ are the measured values of the angles used in computing a_n from b, the angles $A_1, A_2, \ldots$ being opposite to the sides of continuation, then

$$\frac{a_1}{b} = \frac{\sin A_1}{\sin B_1}, \quad \frac{a_2}{a_1} = \frac{\sin A_2}{\sin B_2}, \ldots, \quad \frac{a_n}{a_{n-1}} = \frac{\sin A_n}{\sin B_n}.$$

Hence, by multiplying these expressions together,

$$a_n = b\frac{\sin A_1}{\sin B_1}\,\frac{\sin A_2}{\sin B_2} \cdots \frac{\sin A_n}{\sin B_n}. \tag{1}$$

We may now proceed in two ways.

(*a*) Differentiating directly,

$$da_n = a_n \sin 1''[\cot A\,(A) - \cot B\,(B)],$$

where $(A), (B), \ldots$ denote the corrections to $A, B, \ldots$

[In a chain of triangles it is convenient to use the notation $(A), (B), \ldots$ for $v_1, v_2, \ldots$ the parentheses indicating corrections.]

The condition equations, from the closure of the triangles, are

$$\begin{aligned} (A_1) + (B_1) + (C_1) &= l', \\ (A_2) + (B_2) + (C_2) &= l'', \\ \cdots\cdots\cdots \end{aligned} \tag{2}$$

Substituting in Eq. 7, Art. 124,

$$u_{a_n} = \tfrac{2}{3}\, a_n^2 \sin^2 1'' \,[\cot^2 A + \cot^2 B + \cot A \cot B], \tag{3}$$

the result required.

If the triangles are equilateral, this reduces to

$$u_{a_n} = \tfrac{2}{3}\, n b^2 \sin^2 1''. \tag{4}$$

Hence in a chain of equilateral triangles the weights of the sides decrease as we proceed from the base, b, through the successive triangles, inversely as the number of triangles passed over; that is, are as the fractions

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

(b) Taking logs of both members of Eq. 1. and differentiating,

$$d \log a_n = \frac{d}{dA_1} \log \sin A_1 (A_1) - \frac{d}{dB_1} \log \sin B_1 (B_1) + \cdots$$
$$= [\delta_A (A) - \delta_B (B)], \qquad (5)$$

or expanding the first member,

$$\delta_{a_n} da_n = [\delta_A (A) - \delta_B (B)], \qquad (6)$$

where δ_a is the tabular difference for one unit for the number a_n, and δ_A, δ_B, are the logarithmic differences corresponding to 1″ for the angles A, B, in a table of log sines.

Hence, attending to the condition equations 2, we have from (7) Art. 124

for Eq. 5, $\qquad u_{\log a_n} = \frac{2}{3} [\delta_A^2 + \delta_A \delta_B + \delta_B^2]$,

and for Eq. 6, $\qquad u_{a_n} = \frac{2}{3} \frac{1}{\delta_{a_n}^2} [\delta_A^2 + \delta_A \delta_B + \delta_B^2]$,

as giving the weight of the logarithm of the side and the weight of the side respectively.

Of the two forms (a) and (b), the logarithmic is in general the more convenient in practice.

Compare the final forms derived here with those shown in Ex. 3, p. 164.

Ex. 10. — From a base AB ($= b$) proceeds a chain of equilateral triangles, all of the angles being equally well measured, and the sides BC, CD, . . . being in the same straight line. Find the m. s. e. of the line BN, which is n times the base.

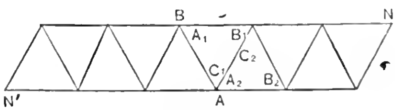

Fig. 6.

Take first the simple case of $n = 2$.

$$F = BN = b \frac{\sin C_1}{\sin B_1} + b \frac{\sin A_1 \sin A_2 \sin C_3}{\sin B_1 \sin B_2 \sin B_3}.$$

$$\therefore dF = \{\cot A_1 (A_1) - 2 \cot B_1 (B_1) + \cot C_1 (C_1)$$
$$+ \cot A_2 (A_2) - \cot B_2 (B_2)$$
$$- \cot B_3 (B_3) + \cot C_3 (C_3)\} b \sin 1''.$$

Also, we have the condition equations

$$(A_1) + (B_1) + (C_1) = l',$$
$$(A_2) + (B_2) + (C_2) = l'',$$
$$(A_3) + (B_3) + (C_3) = l'''.$$

Hence
$$[aa] = 3, \qquad [af] = 0,$$
$$[bb.1] = 3, \qquad [bf.1] = 0,$$
$$[cc.2] = 3, \qquad [cf.2] = 0.$$
$$[ff] = (\cot^2 A_1 + 4\cot^2 B_1 + \cot^2 C_1 + \cot^2 A_2 + \cot^2 B_2 + \cot^2 B_3 + \cot^2 C_3)\, b^2 \sin^2 1''$$
$$= \tfrac{10}{3} b^2 \sin^2 1'', \text{ since } \cot^2 60^\circ = \tfrac{1}{3}.$$

Substituting in Eq. 7, $\quad u_{BN} = \frac{10}{3} b^2 \sin^2 1''$,

and therefore, $\quad \mu_{BN} = \mu \sqrt{\frac{10}{3}}\, b \sin 1''$,

where μ is the m. s. e. of an observed angle.

Generally,
$$\begin{aligned} dF = (n-1)\cot A_1 (A_1) - n \cot B_1 (B_1) + \cot C_1 (C_1) \\ + (n-1)\cot A_2 (A_2) - (n-1)\cot B_2 (B_2) \\ - (n-1)\cot B_3 (B_3) + \cot C_3 (C_3) \\ + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

and
$$u_{BN} = b^2 \sin^2 1'' \left(\frac{4n^3 - 3n^2 + 5n}{9} \right)$$

If the chain proceeds in the opposite direction until $AN' = BN$, then, since $\mu_{AN'}{}^2 = \mu_{BN}{}^2$, and $NN' = 2bn$ approximately, we have
$$\mu_{NN'}{}^2 = \mu NN' \sin 1'' \sqrt{\frac{4n^2 - 3n + 5}{18n}}.$$

If NN' is n times the base $\left(\text{putting } n = \frac{n}{2}\right)$
$$\mu_{NN'} = \mu NN' \sin 1'' \sqrt{\frac{2n^2 - 3n + 10}{18n}}.$$

Hence it follows that in a chain of equilateral triangles where one base only is measured, it is better to place the base at the center of the chain rather than at either end.

Ex. 11. — If two similar isosceles triangles on opposite sides of the base AC are measured independently, thus forming a rhombus (vertices B, B'), then, taking the weight of each angle unity,
$$\mu_{BB'} = \sqrt{2}\, \frac{\mu b \sin 1''}{\sqrt{24}} \operatorname{cosec}^2 \frac{B}{2}$$

and if BB' is n times the base b, then, since $\cot \frac{B}{2} = n$,
$$\mu_{BB'} = \frac{\mu BB' \sin 1''}{2\sqrt{3}} \left(n + \frac{1}{n} \right).$$

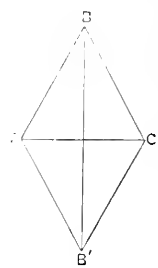

Fig. 7.

Caution. — If we solved for the rhombus directly, it would not do to take
$$BB' = b \cot \frac{B}{2},$$

and then form $\mu_{BB'}$. The result would be $\sqrt{2}$ times too great. For as the triangles are measured independently, each half of BB' must be considered separately, so that we must use the form

$$BB' = \frac{b}{2}\left(\cot\frac{B}{2} + \cot\frac{B'}{2}\right),$$

with the condition equations

$$(A\ ) + (B\ ) + (CC\ ) = l_1,$$
$$(A') + (B') + (CC') = l_2,$$

corresponding to the angles of the two triangles.

Solution in Two Groups.

127. In geodetic work it often happens that the observed quantities are subject to a simple set of conditions which may be readily solved as observation equations by the method of independent unknowns, and are also subject to other conditions which are best solved by the method of correlates. The equations are thus divided into two groups for solution, and the complete solution, therefore, consists of two parts. The observation equations forming the first group are solved by themselves and give approximations to the final values of the unknowns. The corrections to these approximate values due to the second group are next found by solving this second group by the method of correlates.

The merit of the method consists in utilizing the work expended in the solution of the first group in determining the additional corrections due to the second group. The solution is rigorous, and, being broken into two parts, is more easily managed than if all the equations had been solved simultaneously.

Let the first group of equations be the observation equations, n in number and containing n_u unknowns ($n > n_u$),

$$\begin{aligned} a_1x + b_1y + \cdots - l_1 &= v_1, \quad \text{weight } p_1, \\ a_2x + b_2y + \cdots - l_2 &= v_2, \quad \quad '' \quad\ \ p_2, \\ \cdots &\cdots \end{aligned} \tag{1}$$

and the second group the condition equations, n_c in number, involving the same unknowns ($n_c < n_u$),

$$\begin{aligned} a'x + a''y + \cdots - l' &= 0, \\ b'x + b''y + \cdots - l'' &= 0, \\ \cdots &\cdots \end{aligned} \tag{2}$$

The most probable values of the unknowns $x, y, \ldots$ are those which are given by the relation

$$[pv^2] = \text{a minimum}. \tag{3}$$

It is required to find them.

The value of an unknown is found in two parts, the first, (x), $(y), \ldots$ arising from the observation equations, and the second, $(1), (2), \ldots$ arising from the condition equations, thus:

$$\begin{aligned} x &= (x) + (1), \\ y &= (y) + (2), \\ &\cdots\cdots\cdots \end{aligned} \tag{4}$$

Now, overlooking for the present the condition equations, and taking the observation equations only, $(x), (y), \ldots$ would be found by solving these equations in the usual way. We have, therefore, reducing all to weight unity for convenience in writing, the normal equations

$$\begin{aligned} [aa](x) + [ab](y) + \cdots &= [al], \\ [ab](x) + [bb](y) + \cdots &= [bl], \\ \cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{5}$$

The solution of these equations gives (see Art. 103)

$$\begin{aligned} (x) &= [\alpha\alpha][al] + [\alpha\beta][bl] + \cdots \\ (y) &= [\alpha\beta][al] + [\beta\beta][bl] + \cdots \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{6}$$

Hence $(x), (y), \ldots$ are known.

To find the condition corrections $(1), (2), \ldots$, eliminate v_1, $v_2, \ldots v_n$ by substituting in the minimum equation, which then becomes,

$$\begin{aligned} [aa]xx + 2[ab]xy + \cdots &- 2[al]x, \\ + \quad [bb]yy + \cdots &- 2[bl]x, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ [ll] = \text{a min.} \end{aligned} \tag{7}$$

This equation is conditioned by equations 2. Thus, the solution is reduced to that already carried out in Art. 119.

Calling $I, II, \ldots$ the correlates of equations 2, we have the correlate equations

$$\begin{array}{l}[aa]x + [ab]y + \cdots - [al] = a'\,I + b'\,II + \cdots \\ [ab]x + [bb]y + \cdots - [bl] = a''I + b''II + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array}$$

These equations, taken with (4) and (5), give the relations

$$\begin{array}{ll}[aa](1) + [ab](2) + \cdots = a'\,I + b'\,II + \cdots = \boxed{1} & \text{suppose} \\ [ab](1) + [bb](2) + \cdots = a''I + b''II + \cdots = \boxed{2} & \text{"} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array} \tag{8}$$

which being of the same form as (5), their solution gives

$$\begin{array}{l}(1) = [\alpha\alpha]\boxed{1} + [\alpha\beta]\boxed{2} + \cdots \\ (2) = [\alpha\beta]\boxed{1} + [\beta\beta]\boxed{2} + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\end{array} \tag{9}$$

or substituting for $\boxed{1}$, $\boxed{2}$, . . . their values from (8),

$$\begin{array}{l}(1) = \text{A}'\,I + \text{B}'\,II + \text{C}'\,III + \cdots \\ (2) = \text{A}''I + \text{B}''II + \text{C}''III + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array} \tag{10}$$

where

$$\begin{array}{l}\text{A}' = [\alpha\alpha]\,a' + [\alpha\beta]\,a'' + \cdots \\ \text{B}' = [\alpha\alpha]\,b' + [\alpha\beta]\,b'' + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\end{array} \tag{11}$$

and are known quantities.

We have, therefore, expressed the corrections (1), (2), . . . in terms of the unknown correlates, I, II, . . . It remains now to find these correlates.

Substituting for x, y, . . . their values from (4) in the condition equations, and

$$\begin{array}{l}a'(1) + a''(2) + \cdots = l_0', \\ b'(1) + b''(2) + \cdots = l_0'', \\ \cdots\cdots\cdots\cdots\cdots\cdots\end{array} \tag{12}$$

where

$$\begin{array}{l}l_0' = l' - a'(x) - a''(y) - \cdots \\ l_0'' = l'' - b'(x) - b''(y) - \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{array} \tag{13}$$

and are, therefore, known quantities, since (x), (y), . . . are known.

Substitute the values of (1), (2), . . . from (10) in (12), and we have the correlate normal equations,

$$
\begin{aligned}
\left[\overline{aA}\right] I + \left[\overline{aB}\right] II + \cdots &= l_0', \\
\left[\overline{aB}\right] I + \left[\overline{bB}\right] II + \cdots &- l_0'',
\end{aligned}
\tag{14}
$$

. . . .

where

$$
\begin{aligned}
\left[\overline{aA}\right] = [\alpha\alpha]\, a'a' &+ [\alpha\beta]\, a'a'' + \cdots \\
+ [\alpha\beta]\, a'a'' &+ [\beta\beta]\, a''a'' + \cdots
\end{aligned}
\tag{15}
$$

. . . .

The solution of equations 14 gives the correlates I, II, . . . Hence the corrections (1), (2), . . . are known. Also, since (x), (y), . . . have been found from (6), the total corrections x, y, . . . are known.

128. In carrying the preceding solution into practice, the following order of procedure will be found convenient:

(*a*) The formation and solution of the observation equations (1).

The partially adjusted resulting values (x), (y), . . . are now to be used.

(*b*) The formation of the condition equations (12).

$$
\begin{aligned}
a'(1) + a''(2) + \cdots &= l_0', \\
b'(1) + b''(2) + \cdots &= l_0'',
\end{aligned}
$$

. . . .

(*c*) The formation of the weight equations (9). They are at once written down from the general solution of the observation equations in (*a*), and are

$$
\begin{aligned}
(1) &= [\alpha\alpha]\,\boxed{1} + [\alpha\beta]\,\boxed{2} + \cdots \\
(2) &= [\alpha\beta]\,\boxed{1} + [\beta\beta]\,\boxed{2} + \cdots
\end{aligned}
$$

. . . .

(*d*) The formation of the correlate equations (8).

$$
\begin{aligned}
\boxed{1} &= a'I + b'II + \cdots \\
\boxed{2} &= a''I + b''II + \cdots
\end{aligned}
$$

. . . .

(*e*) The expression of the corrections in terms of the correlates by substituting from (*d*) in (*c*).

$$(1) = \mathrm{A}'I + \mathrm{B}'II + \cdots$$
$$(2) = \mathrm{A}''I + \mathrm{B}''II + \cdots$$
$$\cdots\cdots\cdots\cdots$$

(*f*) The formation of the normal equations by substituting from (*c*) in (*b*). They are,

$$[a\mathrm{A}]\, I + [a\mathrm{B}]\, II + \cdots = l_0',$$
$$[a\mathrm{B}]\, I + [b\mathrm{B}]\, II + \cdots = l_0'',$$
$$\cdots\cdots\cdots\cdots$$

(*g*) The determination of the corrections by substituting the values of the correlates in (*e*).

129. To Find the Precision of the Adjusted Values or of any Function of them.

(*a*) First find r, the p. e. of an observation of weight unity.

We have (Art. 121) $r = .6745\ \mu$,

$$\mu^2 = \frac{[v^2]}{\text{number of conditions}}$$
$$= \frac{[v^2]}{(n - n_u) + n_c},$$

since $n - n_u$ is the number of conditions in the observation equations, and n_c the number in the condition equations.

To find $[v^2]$. From the first observation equation

$$\begin{aligned} v_1 &= a_1x + b_1y + \cdots - l_1 \\ &= a_1(x) + b_1(y) + \cdots - l_1 + a_1(1) + b_1(2) + \cdots \\ &= v_1{}^0 + a_1(1) + b_1(2) + \cdots \end{aligned}$$

Similarly

$$v_2 = v_2{}^0 + a_2(1) + b_2(2) + \cdots$$
$$\cdots\cdots\cdots\cdots$$

where

$$v_1{}^0 = a_1(x) + b_1(y) + \cdots - l_1,$$
$$v_2{}^0 = a_2(x) + b_2(y) + \cdots - l_2,$$
$$\cdots\cdots\cdots\cdots$$

that is, v_1^0, v_2^0, . . . are the residuals arising from taking the observation equations only.

Attending to Eq. 5, Art. 127, it follows evidently that

$$[av^0] = 0 \quad [bv^0] = 0, \ . \ . \ .$$

Square the residuals v_1, v_2, . . . and add, then

$$\begin{aligned}[v^2] &= [v^0v^0] + [\{a\,(1) + b\,(2) + \cdots\}^2] \\ &= [v^0v^0] + [ww] \text{ suppose.}\end{aligned}$$

The total sum $[v^2]$ may therefore be found in two parts, one from squaring the residuals of the observation equations, and the other from the corrections (1), (2), . . .

We proceed to put $[ww]$ in a more convenient shape for computation.

$$\begin{aligned}[ww] &= [\{a\,(1) + b\,(2) + \cdots\}^2] \\ &= (1)\{[aa]\,(1) + [ab]\,(2) + \cdots\} \\ &+ (2)\{[ab]\,(1) + [bb]\,(2) + \cdots\} \\ &+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ &= (1)\,\boxed{1} + (2)\,\boxed{2} + \cdots\end{aligned}$$

from Eq. 8, Art. 127.

Substitute for (1), $\boxed{1}$, (2), . . . their values from equations 8 and 10, Art. 127, and expand; then

$$\begin{aligned}[ww] = &\left\{\left[\overline{aA}\right] I + \left[\overline{aB}\right] II + \cdots\right\} I \\ &+ \left\{\left[\overline{aB}\right] I + \left[\overline{bB}\right] II + \cdots\right\} II \\ &+ \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ .\end{aligned}$$

which may be transformed, by means of Eq. 14, into the form

$$[ww] = l_0{}'I + l_0{}''II + \cdots$$

or, as in Art. 121, into the form

$$[ww] = \frac{(l_0{}')^2}{\left[\overline{aA}\right]} + \frac{(l_0.1'')^2}{\left[\overline{bB.1}\right]} + \frac{(l_0.2''')^2}{\left[\overline{cC.2}\right]} + \cdots$$

These forms may be readily computed as in Art. 106.

130. (*b*) Next find the weight of the given function of the adjusted values.

Let the function, reduced to the linear form, be

$$dF = g_1 x + g_2 y + \cdots \tag{16}$$

where g_1, g_2, . . . are known quantities.

Put for x, y, . . . their values $(x) + (1)$, $(y) + (2)$, . . . and

$$dF = g_1\,(x) + g_2\,(y) + \cdots + g_1\,(1) + g_2\,(2) + \cdots$$

Put for (1), (2), . . . their values from (10), and

$$dF = g_1\,(x) + g_2\,(y) + \cdots + [g\text{A}]\,I + [g\text{B}]\,II + \cdots \tag{17}$$

where I, II, . . . are found from the equations

$$\begin{aligned} &[\overline{a\text{A}}]\ I + [\overline{a\text{B}}]\ II + \cdots - l_0' = 0, \\ &[\overline{a\text{B}}]\ I + [\overline{b\text{B}}]\ II + \cdots - l_0'' = 0, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

Using the multipliers k_1, k_2, . . . in order to eliminate I, II, . . . , we have, as in Art. 121,

$$\begin{aligned} dF = {} & g_1\,(x) + g_2\,(y) + \cdots + l_0'k_1 + l_0''k_2 + \cdots \\ & + \left\{ [g\text{A}] - [\overline{a\text{A}}]\,k_1 - [\overline{a\text{B}}]\,k_2 - \cdots \right\} I, \\ & + \left\{ [g\text{B}] - [\overline{a\text{B}}]\,k_1 - [\overline{b\text{B}}]\,k_2 - \cdots \right\} II, \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned} \tag{18}$$

We may determine k_1, k_2, . . . so as to cause the coefficients of I, II, . . . to vanish; that is, so as to satisfy the equations

$$\begin{aligned} &[\overline{a\text{A}}]\ k_1 + [\overline{a\text{B}}]\ k_2 + \cdots = [g\text{A}], \\ &[\overline{a\text{B}}]\ k_1 + [\overline{b\text{B}}]\ k_2 + \cdots = [g\text{B}], \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned} \tag{19}$$

and then we shall have

$$dF = g_1\,(x) + g_2\,(y) + \cdots + l_0'k_1 + l_0''k_2 + \cdots$$

Substitute for l_0', l_0'', . . . from (13), and

$$dF = [lk] + G_1\,(x) + G_2\,(y) + \cdots \tag{20}$$

where

$$\begin{aligned} G_1 &= g_1 - a'k_1 - b'k_2 - \cdots \\ G_2 &= g_2 - a''k_1 - b''k_2 - \cdots \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned} \tag{21}$$

We have thus expressed the function in terms of (x), (y), . . . and known quantities.

Now, since (x), (y), . . . are not independent, but are connected by the equations

$$\begin{aligned}
[aa](x) + [ab](y) + \cdots &= [al], \\
[ab](x) + [bb](y) + \cdots &= [bl], \\
\cdots\cdots\cdots
\end{aligned}$$

the problem is reduced to that already solved in Art. 108.

If, therefore, u_F is the reciprocal of the required weight,

$$u_F = [GQ] \tag{22}$$

where

$$\begin{aligned}
Q_1 &= [\alpha\alpha]\, G_1 + [\alpha\beta]\, G_2 + \cdots \\
Q_2 &= [\alpha\beta]\, G_1 + [\beta\beta]\, G_2 + \cdots \\
\cdots\cdots\cdots
\end{aligned} \tag{23}$$

the quantities $[\alpha\alpha]$, $[\alpha\beta]$, . . . being as in the weight equations 9.

Putting for G_1, G_2, . . . their values from (21) in these equations, and attending to (11), we find

$$\begin{aligned}
Q_1 &= q_1 - \text{A}' k_1 - \text{B}' k_2 - \cdots \\
Q_2 &= q_2 - \text{A}'' k_1 - \text{B}'' k_2 - \cdots \\
\cdots\cdots\cdots
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
q_1 &= [\alpha\alpha]\, g_1 + [\alpha\beta]\, g_2 + \cdots \\
q_2 &= [\alpha\beta]\, g_1 + [\beta\beta]\, g_2 + \cdots \\
\cdots\cdots\cdots
\end{aligned} \tag{25}$$

Substituting in (22) for G_1, G_2, . . . Q_1, Q_2, . . . their values from (21) and (24),

$$\begin{aligned}
[GQ] = [gq] &- [g\text{A}]\, k_1 - [g\text{B}]\, k_2 - \cdots \\
&- [aq]\, k_1 - [bq]\, k_2 - \cdots \\
&+ \left\{ [\overline{a\text{A}}]\, k_1 + [\overline{a\text{B}}]\, k_2 + \cdots \right\} k_1 \\
&+ \left\{ [\overline{a\text{B}}]\, k_1 + [\overline{b\text{B}}]\, k_2 + \cdots \right\} k_2 \\
&+ \cdots\cdots\cdots
\end{aligned}$$

But from (11) and (25)

$$[aq] = [g\text{A}],\ [bq] = [g\text{B}],\ \ldots$$

Hence, attending to (19), the above expression reduces to

$$[GQ] = [gq] - [g\text{A}]\,k_1 - [g\text{B}]\,k_2 - \cdots$$

or to

$$[GQ] = [gq] - \frac{[g\text{A}]^2}{[a\text{A}]} - \frac{[g\text{B}.1]^2}{[b\text{B}.1]} - \frac{[g\text{C}.2]^2}{[c\text{C}.2]} - \cdots$$

To compute $[gq]$. Multiply each of equations 25, in order, by $g_1, g_2, \ldots$, and add, and

$$\begin{aligned}[gq] = [\alpha\alpha]\,g_1g_1 &+ 2\,[\alpha\beta]\,g_1g_2 + \cdots \\ &+ \quad [\beta\beta]\,g_2g_2 + \cdots \\ &\qquad\qquad\quad + \cdots\end{aligned}$$

where $[\alpha\alpha]$, $[\alpha\beta]$, . . . may be taken from the weight equations.

The remaining terms of the second form of $[GQ]$ may be found from the solution of the normal equations, as shown in Art. 121.

Solution by Successive Approximation.

131. This method of solution (due to Gauss) is of the greatest importance in adjustments involving many conditions. It may be stated as follows:

The condition equations may be divided into groups, and the groups solved in any order we please. Each successive group will give corrections to the values furnished by the preceding groups, and the corrected values will be closer and closer approximations to the most probable values which would be found from the simultaneous solution of all the groups.

For suppose we have the condition equations

$$\begin{aligned} a'v_1 + a''v_2 + \cdots &= l_a, \\ b'v_1 + b''v_2 + \cdots &= l_b, \\ \cdots\cdots & \\ h'v_1 + h''v_2 + \cdots &= l_h, \\ k'v_1 + k''v_2 + \cdots &= l_k, \\ \cdots\cdots & \end{aligned}$$

with

$$[pv^2] = \text{a min.}$$

Let v_1', v_2', . . . be the values of v_1, v_2, . . . obtained from solving the first group alone; that is, from

$$a'v_1' + a''v_2' + \cdot \cdot \cdot = l_a,$$
$$b'v_1' + b''v_2' + \cdot \cdot \cdot = l_b,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$[pv'^2] = \text{a min.}$$

If now (v_1'), (v_2'), . . . are the corrections to these values resulting from the remaining equations, then since

$$\begin{aligned}[pv^2] &= [p\,\{v' + (v')\}^2] \\ &= [pv'^2] + [p\,(v')^2],\end{aligned}$$

the condition equations are reduced to

$$a'\,(v_1') + a''\,(v_2') + \cdot \cdot \cdot = l_a',$$
$$b'\,(v_1') + b''\,(v_2') + \cdot \cdot \cdot = l_b',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$h'\,(v_1') + h''\,(v_2') + \cdot \cdot \cdot = l_h',$$
$$k'\,(v_1') + k''\,(v_2') + \cdot \cdot \cdot = l_k',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

with

$$[p\,(v')^2] = \text{a min.},$$

and the values of (v') found from the simultaneous solution of these equations, added to the values of v' found from the solution of the first set, would be equal to the value of v found directly.

Similarly, if v_1'', v_2'', . . . be the values of (v') obtained by solving the second set alone, and (v_1''), (v_2''), . . . be the corrections to these values resulting from the remaining equations, then since

$$[pv^2] = [pv'^2] + [pv''^2] + [p\,(v'')^2]$$

the condition equations are reduced to

$$a'\,(v_1'') + a''\,(v_2'') + \cdot \cdot \cdot = l_a'',$$
$$b'\,(v_1'') + b''\,(v_2'') + \cdot \cdot \cdot = l_a'',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$h'\,(v_1'') + h''\,(v_2'') + \cdot \cdot \cdot = l_h'',$$
$$k'\,(v_1'') + k''\,(v_2'') + \cdot \cdot \cdot = l_k'',$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

with

$$[p\,(v'')^2] = \text{a min.}$$

The quantities $[pv'^2]$, $[pv''^2]$, . . . being positive, the minimum equation is reduced with the solution of each set, and thus we gradually approach the most probable set of values. Beginning with the first set a second time, and solving through again, we should reduce the minimum equation still farther, and by continuing the process we shall finally reach the same result as that obtained from the rigorous solution. In practice the first approximation is in general close enough.

It is plain that the most probable values can be found after any approximation by solving simultaneously the whole of the groups, using the values already found as approximations to these most probable values.

Examples will be found in the next chapter.

CHAPTER VI

APPLICATION TO THE ADJUSTMENT OF A TRIANGULATION. METHOD OF ANGLES

132. The adjustment of the measured angles of a triangulation net is a special case of the problem discussed in the preceding chapters. We assume the reader to be acquainted with the construction and method of handling of instruments used in measuring horizontal angles, and shall confine ourselves to the methods of adjusting the measured values of the angles.

133. For clearness we will explain in some detail the preliminary work necessary for the formation of the condition equations. In a triangulation there must be one measured base at least, as AB. Starting from this base, and measuring the angles CAB, ABC, we may compute the sides, AC, BC by the ordinary rules of trigonometry. In plotting the figure, the point C can be located in but one way, as only the measurements necessary for this purpose have been made.

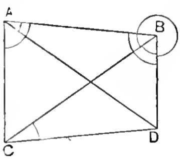

Fig. 8.

Similarly, by measuring the angles CBD, DCB we may plot the position of the point D, and this can be done in but one way. If, however, the observer, while at A, had also read the angle DAB, then the point D could have been plotted in two ways, and we should find in almost all cases that the lines AD, BD, CD would not intersect in the same point. In other words, in computing the length of a side from the base we should find different values, according to the triangles through which we passed. Thus the value of CD computed from AB would not, in general, be the same if found from the triangles ABC, BCD, and from ABC, CAD.

If the exterior angle ABD had also been measured, we should have another contradiction, arising from the non-satisfaction of the relation

$$DBC + CBA + ABD = 360°.$$

And not these contradictions only. For we have considered so far that in a triangle, only two of the angles are measured. If in the first triangle, ABC, the third angle, BCA, were also measured, we know from spherical geometry that the three angles should satisfy the relation

$$CAB + ABC + BCA = 180° + \text{sph. excess of triangle},$$

which the measured values will not do in general. A similar discrepancy may be expected in the other triangles.

In a triangulation net, then, with a single measured base, in which the sides are to be computed from this base through the intervening triangles, we conclude that the contradictions among the measured angles may be removed and a consistent figure obtained if the angles are adjusted so as to satisfy the two classes of conditions:

(1) Those arising at each station from the relations of the angles to one another at that station.

These are known as *local* conditions.

(2) Those arising from the geometrical relations necessary to form a closed figure.

(*a*) That the sum of the angles of each triangle in the figure should be equal to 180° increased by the spherical excess of the triangle.

(*b*) That the length of any side, as computed from the base, should be the same whatever route is chosen.

These are known as *general* conditions.

134. The number of conditions to be satisfied will depend on the measurements made. Each condition can be stated in the form of an equation in which the most probable values of the measured quantities are the unknowns. The number of equations being less than the number of unknowns, an infinite

number of solutions is possible. The problem before us is to select the most probable values from this infinite number of possible values.

The general statement of the method of solution is this. Adjust the angles so as to satisfy simultaneously the local and general conditions; that is, of all possible systems of corrections to the observed quantities which satisfy these conditions, to find that system which makes the sum of the squares of the corrections a minimum.

The form of the reduction depends on the methods employed in making the observations. These methods, in general terms, are as follows: Let O in the figure be the station occupied, and A, B, C signals sighted at. The angles AOB, BOC are required. By pointing at A and then at B we find the angle AOB. Point now at B and next at C, and we have the angle BOC. These two angles are *independent* of one another.

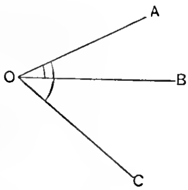

Fig. 9.

If, however, we had pointed at A, B, C in succession we should also have found the angles AOB, BOC, but they would not be independent of one another, as the reading to B enters into each.

The first method of measurement is known as the method of independently measured angles, or, if each angle is mechanically multiplied before being read off from the circle, it is called the method of repetition; and the second method is known as the method of directions.

The Method of Independent Angles.

135. As the case of independent angles is the simplest to reduce, we shall begin with it.

A distinction must be made between angles that are independently observed and angles which are independent in the sense that no condition exists between them. Thus at the

station O, above, the angles AOB, BOC, AOC might be observed independently of one another, but we should not call them independent angles, since the condition

$$AOC = AOB + BOC$$

must be satisfied between them. By independent angles, therefore, in the reduction, we mean those measured angles in terms of which all the measured angles can be expressed by means of the conditions connecting them. In the present case any two of the three angles AOB, BOC, AOC may be taken as independent, and the third angle would be dependent.

Angles may be measured independently either with a repeating or with a non-repeating theodolite. In primary work a non-repeating theodolite in which the graduated limb is read by microscopes furnished with micrometers is to be preferred. The method of reading an angle is as follows: The instrument, having been carefully adjusted, is directed to the left-hand signal and the micrometers read. It is then directed to the other signal and the micrometers again read. The difference between these readings is called a positive single result. The whole operation is repeated in reverse order; that is, beginning with the second signal and ending with the first, giving a negative single result. The mean of these two results is called a combined result, and is free from the error arising from uniform twisting of the post or tripod on which the instrument is placed, or from "twist of station," as it is called, provided the rate of observing is constant.

The telescope is next turned 180° in azimuth and then 180° in altitude, leaving the same pivots in the same wyes, and another combined result is obtained. The mean of the two combined results is free from errors of the instrument arising from imperfect adjustments for collimation, from inequality in the heights of the wyes, and from inequality of the pivots.

The distinction between these two combined results is noted in the record by "telescope direct" and "telescope reverse."

136. Besides those mentioned, there are two kinds of systematic error in measuring angles that deserve special attention. They are the errors arising from the regular or "periodic" errors of graduation of the horizontal limb of the instrument, and the error from the inclination of the limb itself to the horizon. The effects of the first may be got rid of by the method of observation, as follows: The reading of the limb on the first signal is changed (usually after each pair of combined results) by some aliquot part of the distance, or half-distance, between consecutive microscopes in case of two-microscope and three-microscope instruments respectively. Thus, if n is the number of pairs of combined results desired, the changes would be $\frac{180}{n}$ and $\frac{60°}{n}$ respectively with the instruments mentioned. The operation of reversal in case of a three-microscope instrument causes each microscope to fall at the middle of the opposite 120° space, the limb remaining unchanged. Thus, if the full lines in Fig. 10 represent the positions of the microscopes with telescope direct, the dotted lines show their positions with telescope reverse. In this lies the greatest advantage of three microscopes over two, since with the latter, in reversing, the microscopes simply change places with each other, without reading on new portions of the limb.

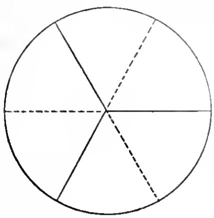

Fig. 10.

The error arising from want of level of the horizontal limb cannot be eliminated by the method of observation, but with the levels which accompany a good instrument, and with ordinary care, it will usually be less than 0.1″. In case, however, of a signal having a high altitude above the horizon, the error from this source may be greater, and then special care should be taken in leveling. For an expression for its influence in any case, see Chauvenet's *Astronomy*, Vol. II, Art. 211.

It is desirable to make the observations under various conditions so as to avoid constant errors. See Appendix No. 4, U. S. C. & G. Survey Report for 1903, pp. 843–844, 869.

137. We shall for illustration take the following example, making use of such parts of it from time to time as may belong to the subject in hand, and finally, after explaining the method of forming the condition equations, solve it in full.

In the triangulation of Lake Superior executed by the U. S. Engineers the following angles were measured in the quadrilateral N. Base, S. Base, Lester, Oneota.

$LNO =$	$124^\circ\ 09'\ 40.69''$	weight	2
$SNL =$	$113^\circ\ 39'\ 05.07''$	"	2
$ONS =$	$122^\circ\ 11'\ 15.61''$	"	14
$NSO =$	$23^\circ\ 08'\ 05.26''$	"	23
$LSN =$	$47^\circ\ 31'\ 20.41''$	"	6
$LSO =$	$70^\circ\ 39'\ 24.60''$	"	7
$SON =$	$34^\circ\ 40'\ 39.66''$	"	31
$NOL =$	$43^\circ\ 46'\ 26.40''$	"	1
$OLS =$	$30^\circ\ 53'\ 30.81''$	"	8

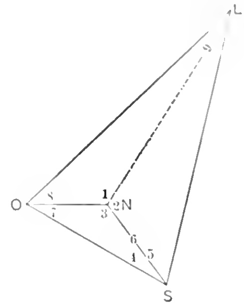

Fig. 11.

These angles we shall denote by M_1, M_2, . . . , M_9 respectively.

The length of the line N. Base — S. Base (Minnesota Point) is 6056.6 m., and the latitudes of the four stations are approximately

N. Base, $46^\circ\ 45'$	Lester, $46^\circ\ 52'$
S. Base, $46^\circ\ 43'$	Oneota, $46^\circ\ 45'$

138. The Local Adjustment. — When in a system of triangulation the horizontal angles read at a station are adjusted for all of the conditions existing among them, then these angles are said to be locally adjusted.

From the considerations set forth in Art. 133, it is readily seen that at a station only two kinds of conditions are possible:

(*a*) that an angle can be formed from two or more others, and

(*b*) that the sum of the angles round the horizon should be equal to 360°.

The second of these is included in the first, and the method of adjustment may be stated in general terms as follows:

An inspection of the figure representing the angles at the station will show how all of the measured angles can be expressed in terms of a certain number of them which are independent of one another. These relations will give rise to condition equations, or *local equations*, as they are called, which may be solved as in Chapters IV or V.

Thus, if $M_1, M_2, \ldots M_n$ denote the single measured angles, and $v_1, v_2, \ldots v_n$ their most probable corrections, then if any of the angles M_h, M_k can be formed from others, we have, by equating the measured and computed values, the local condition equations,

$$M_h + v_h = M_1 + v_1 + M_2 + v_2 + \cdots$$
$$M_k + v_k = M_1 + v_1 + M_2 + v_2 + \cdots$$
$$\cdots\cdots\cdots\cdots$$

or

$$v_1 + v_2 + \cdots - v_h = l_h \text{ suppose,}$$
$$v_1 + v_2 + \cdots - v_k = l_k \text{ suppose,}$$
$$\cdots\cdots\cdots\cdots$$

with

$$p_1v_1^2 + p_2v_2^2 + \cdots + p_hv_h^2 + p_kv_k^2 + \cdots + p_nv_n^2 = \text{a minimum}$$

where $p_1, p_2, \ldots p_n$ denote the weights of the angles.

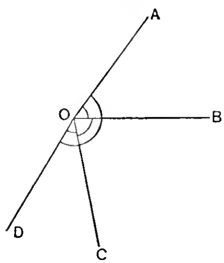

Fig. 12.

The solution may be in general best carried out by the method of correlates, as in Chapter V.

139. The following special cases are of frequent occurrence:

(1) At a station O the $n - 1$ single angles AOB, BOC, . . . are measured, and also the sum angle AOL, to find the adjusted values of the separate angles, all of the measured values being of the same weight.

The condition equation is

$$M_1 + v_1 + M_2 + v_2 + \cdots + M_{n-1} + v_{n-1} = M_n + v_n,$$

or

$$v_1 + v_2 + \cdots + v_{n-1} - v_n = M_n - (M_1 + M_2 + \cdots + M_{n-1})$$
$$= l \text{ suppose,}$$

with

$$[v^2] = \text{a minimum.}$$

The solution gives (Art. 118 or 119),

$$v'_1 = v'_2 = \cdots = - v'_n = \frac{l}{n},$$

that is, *the correction to each angle is* $\frac{1}{n}$ *of the excess of the sum angle over the sum of the single angles, and the sign of the correction to the sum angle is opposite to that of the single angles.*

(2) At a station O the n single angles AOB, BOC, . . . LOA are measured, thus closing the horizon, to find the adjusted values of the angles.

The condition equation is

$$v'_1 + v'_2 + \cdots + v'_n = 360^\circ - (M_1 + M_2 + \cdots + M_n)$$
$$= l \text{ suppose,}$$

with $$[pv^2] = \text{a minimum.}$$

The solution gives

$$v'_1 = u_1 \frac{l}{[u]},$$

$$v'_2 = u_2 \frac{l}{[u]},$$

$$\cdots\cdots$$

where $$u_1 = \frac{1}{p_1}, \quad u_2 = \frac{1}{p_2}, \quad \ldots \quad \text{and } [u] = \left[\frac{1}{p}\right].$$

If the weights are equal, then

$$v_1 = v'_2 = \cdots = v'_n = \frac{l}{n};$$

that is, *the correction to each angle is* $\frac{1}{n}$ *of the excess of* 360° *over the sum of the measured angles.*

Ex. 1. — The angles at Station N. Base close the horizon; required to adjust them.

We have (Art. 137),

$$M_1 + v'_1 = 124^\circ \; 09' \; 40.69'' + v'_1 \quad \text{weight } 2$$
$$M_2 + v'_2 = 113^\circ \; 39' \; 05.07'' + v'_2 \quad \text{"} \quad 2$$
$$M_3 + v'_3 = 122^\circ \; 11' \; 15.61'' + v'_3 \quad \text{"} \quad 1.4$$

Sum . . . . $= 360^\circ \; 00' \; 01.37'' + v'_1 + v'_2 + v'_3$

Theoretical sum . . $= 360^\circ \; 00' \; 00.00''$

$\therefore$ Local equation is . $= \quad 0^\circ \quad = \quad 1.37'' + v'_1 + v'_2 + v'_3$

Hence (Ex. 2, Art. 120),
$$\begin{aligned} v_1 &= -\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}+\frac{1}{14}} \times 1.37 \\ &= -0.64'', \\ v_2 &= -0.64'', \\ v_3 &= -0.09'', \end{aligned}$$

and the adjusted angles are,

$$\begin{array}{rrrr} & 124^\circ & 09' & 40.05'' \\ & 113^\circ & 39' & 04.43'' \\ & 122^\circ & 11' & 15.52'' \\ \hline \text{Check-sum} = & 360^\circ & 00' & 00.00'' \end{array}$$

Ex. 2. — Precisely as in the preceding we may deduce at Station S. Base the local equation,

$$0 = 1.07'' + v_4 + v_5 - v_6,$$

and the adjusted angles

$$\begin{array}{rrr} 23^\circ & 08' & 05.13'', \\ 47^\circ & 31' & 19.91'', \\ 70^\circ & 39' & 25.04''. \end{array}$$

140. Number of Local Equations at a Station.—If s stations are sighted at from a station that is occupied, the number of angles necessary to be measured to determine all of the angles that can be formed at the station occupied is $s - 1$. If, therefore, an additional angle were measured, its value could be determined in two ways: from the direct measurement and from the $s - 1$ measures. The contradiction in these two values would give rise to a local (condition) equation. If, therefore, n is the total number of angles measured at a station, the number of local equations, as indicated by the number of superfluous angles, is

$$n - s + 1.$$

141. The General Adjustment. — With a single measured base, the number of conditions arising from the geometrical relations existing among the different parts of a triangulation net can be readily estimated. For if the net contains s stations, two are known, being the end points of the base, and $s - 2$ are to be found.

Now, two angles observed at the end points of the base will determine a third point; two more observed at the end points of a line joining any two of these points will determine a fourth

point, and so on. Hence, to determine the $s - 2$ points, $2(s - 2)$ angles are necessary. If, therefore, n is the total number of locally independent angles, the number of superfluous angles, that is, the number of conditions to be satisfied, is

$$n - 2(s - 2).$$

Ex. — In a chain of triangles, if s is the number of stations, show that the number of conditions to be satisfied is $s - 2$; and in a chain of quadrilaterals, with both diagonals drawn, the number of conditions is $2s - 4$.

The equations arising from these conditions are divided into two classes, angle equations and side equations.

142. The Angle Equations. — The sum of the angles of a triangle drawn on a plane surface is equal to 180°. The sum of the angles of a spherical triangle exceeds 180° by the spherical excess (ϵ) of the triangle, which latter is found from the relation

$$\epsilon = \frac{\text{area of triangle}}{r^2 \sin 1''},$$

r being the radius of the sphere.

From surveys carried on during the past two centuries, the earth has been found to be spheroidal in form, and its dimensions have been determined within small limits. Now, a spheroidal triangle of moderate size may be computed as a spherical triangle on a tangent sphere whose radius is $\sqrt{RN}$, where R, N, are the radii of curvature of the meridian and of the normal section to the meridian respectively at the point corresponding to the mean of the latitudes ϕ of the triangle vertices.

Hence we may wrap our triangulation on the spheroid in question by conforming it to the spherical excess computed from the formula,

$$\epsilon \text{ (in seconds)} = \frac{a_1 b_1 \sin C_1}{2 RN \sin 1''},$$

where a_1, b_1, are two sides and C_1 is the included angle of the triangle.

For convenience of computation we may write,

$$\epsilon = m a_1 b_1 \sin C_1,$$

when log m may be tabulated for the argument ϕ. The following table is computed with Clarke's values of the elements of the terrestrial spheroid of 1866 corresponding to latitudes from 10° to 70°. The meter is the unit of length to be used.

Table of log m.

Latitude.	Log m.	Latitude.	Log m.	Latitude.	Log m.	Latitude.	Log m.
° ′		° ′		° ′		° ′	
18 00	$\bar{1}.40639$	33 00	$\bar{1}.40520$	48 00	$\bar{1}.40369$	63 00	$\bar{1}.40227$
18 30	636	33 30	516	48 30	364	63 30	223
19 00	632	34 00	511	49 00	359	64 00	219
19 30	629	34 30	506	49 30	354	64 30	215
20 00	626	35 00	501	50 00	349	65 00	210
20 30	623	35 30	496	50 30	344	65 30	207
21 00	619	36 00	491	51 00	339	66 00	203
21 30	616	36 30	486	51 30	334	66 30	199
22 00	612	37 00	482	52 00	329	67 00	195
22 30	608	37 30	477	52 30	324	67 30	192
23 00	605	38 00	472	53 00	319	68 00	188
23 30	601	38 30	467	53 30	314	68 30	185
24 00	597	39 00	462	54 00	309	69 00	181
24 30	594	39 30	457	54 30	304	69 30	178
25 00	590	40 00	452	55 00	299	70 00	174
25 30	586	40 30	446	55 30	295	70 30	171
26 00	582	41 00	441	56 00	290	71 00	168
26 30	578	41 30	436	56 30	285	71 30	164
27 00	573	42 00	431	57 00	280	72 00	$\bar{1}.40161$
27 30	569	42 30	426	57 30	276	....	.....
28 00	565	43 00	421	58 00	271	....	.....
28 30	560	43 30	416	58 30	266	....	.....
29 00	556	44 00	411	59 00	262	....	.....
29 30	552	44 30	406	59 30	257	....	.....
30 00	548	45 00	400	60 00	253	....	.....
30 30	544	45 30	395	60 30	249	....	.....
31 00	539	46 00	390	61 00	244	....	.....
31 30	534	46 30	385	61 30	240	....	.....
32 00	530	47 00	380	62 00	235	....	.....
32 30	$\bar{1}.40525$	47 30	$\bar{1}.40375$	62 30	$\bar{1}.40231$	....	.....

To find a_1, b_1, ϕ, a preliminary geodetic computation must first be made of the triangulation to be adjusted, starting from a base or from a known side. The values found from using the unadjusted angles will be close enough for this purpose. An error of less than 3′ in the latitude will not under any circumstances produce an error of over 0.001″ in the computed spherical excess, and in general therefore the latitudes may be taken from a map or sketch of the triangulation.

143. A useful check of the excess results from the principle that the sums of the excesses of triangles that cover the same area should be equal. In our example the spherical excesses of the triangles *ONS*, *LSO* will be found to be 0.05″ and 0.37″ respectively.

In each single triangle, then, the condition required to wrap it on the spheroid, that is, that the sum of the three measured angles shall be equal to 180°, together with the spherical excess, gives a condition equation.* This is called an *angle equation*, or by some a *triangle equation*.

Ex. — In the triangle N. Base, S. Base, Oneota, if v_3, v_4, v_7, denote the corrections to the three angles, we have for the most probable values,

$$
\begin{array}{lrrrl}
 & ONS = & 122^\circ & 11' & 15.61'' + v_3 \\
 & NSO = & 23^\circ & 08' & 05.26'' + v_4 \\
 & SON = & 34^\circ & 40' & 39.66'' + v_7 \\
\hline
\text{Sum} \;\; . \; . \; . \; . & = & 180^\circ & 00' & 00.53'' + v_3 + v_4 + v_7 \\
\text{Theoretical sum} & = & 180^\circ & 00' & 00.05'' = 180^\circ + \epsilon
\end{array}
$$

and the angle equation is formed by equating these sums. The result is,

$$v_3 + v_4 + v_7 + 0.48'' = 0.$$

Similarly, from the triangle Lester, S. Base, Oneota, the angle equation is,

$$v_6 + v_7 + v_8 + v_9 + 1.10'' = 0.$$

144. Number of Angle Equations in a Net. — It is to be expected that in a triangulation net some of the lines will be

* We confine ourselves throughout to triangles to which Legendre's theorem is applicable. For very large triangles other formulas for spherical excess must be used if great accuracy is required. See *The Transcontinental Triangulation of C. & G. Survey*, pp. 51–54.

sighted over in both directions, and some only in one direction. If these latter lines are omitted, the number of angle equations will remain unaltered. Thus in our Lake Superior quadrilateral (Fig. 11) the line NL has been sighted over from N, but not from L, so that we have only two angle equations: namely, those resulting from the triangles ONS, OLS, just as if the figure had been of the form of Fig. 13, in which the line NL is omitted.

Let s be the total number of stations in a figure or series of figures regardless of whether the stations are occupied or not, s_u the number of unoccupied stations, l the total number of lines in the figure, and l_1 the number of lines which are observed over in one direction only. The number of angle equations will be $l - l_1 - s + s_u + l$.

This may be proved by plotting the figure in detail, adding at each step of the process one new point and all observed lines connecting that point with points already shown on the figure.

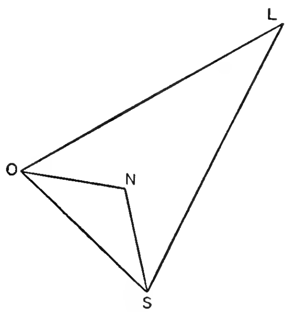

Fig. 13.

The formula is true in any case, because, as the figure is drawn point by point and line by line as indicated, it holds for the simplest possible figure the triangle; for the first two lines to any new occupied point, $l - s$ is increased by one and the value of no other symbol is changed in the formula and one new angle equation appears; for each new complete line to a new occupied point after the first two, l is increased by one, and one new angle equation appears; for each new line observed in one direction only to a new point after the first two lines to it are drawn, $l - l_1$ remains unchanged, and no new angle equation appears; the addition of an unoccupied station with any number of lines to it which are necessarily observed in one direction only does not change the value of $l - l_1 - s + s_u + l$, and no new angle equation is introduced.

It may seem that there are more angle equations than have been indicated, but it will be found in every such case that the supposed additional equation may be derived algebraically from those already used, and is therefore not a new independent equation.

Ex. — In the quadrilateral *ABCD*, in which all of the 8 angles are measured, show that there are three independent angle equations, and that these may be found from the following 8 sets of figures:

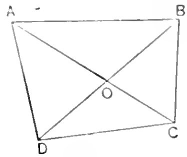

Fig. 14.

ABD, *ABC*, *ACD*; *ABD*, *ABC*, *ABCD*;
ABD, *ACD*, *ABCD*;
BDA, *BCD*, *BCA*; *BCA*, *BCD*, *BCDA*;
CDB, *CAB*, *CDA*; *CDB*, *CDA*, *CDBA*;
DAB, *DBC*, *DAC*.

145. The Side Equations. — In a single triangle, or in a simple chain of triangles, the length of any assigned side can be computed from a given side in but one way. When the triangles are interlaced, this is not the case.

Thus in Fig. 13 any side can be computed from *NS* in but one way. The only condition equations apart from the local equations would be the two angle equations. But in Fig. 11, in which the line *NL* is sighted over from *N*, we have the further condition that the lines *OL*, *NL*, *SL* intersect in the same point, *L*. The figure plotted from the measured values would be of the form of Fig. 15.

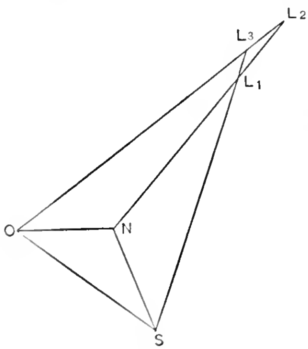

Fig. 15.

To express in the form of an equation the condition that the three points L_1, L_2, L_3 must coincide, we proceed as follows: Starting from the base *NS*, we may compute SL_1 directly from the triangle SNL_1, and SL_3 from the triangles *SON*, SOL_3. This gives

$$\frac{\text{side } SN}{\text{side } SL_1} = \frac{\sin SL_1N}{\sin SNL_1},$$

$$\frac{\text{side } SN}{\text{side } SL_3} = \frac{\sin SON \sin SL_3O}{\sin SNO \sin SOL_3};$$

but SL_1 must be equal to SL_3.

Hence the condition equation is

$$\frac{\sin SLN}{\sin SNL} \frac{\sin SNO}{\sin SON} \frac{\sin SOL}{\sin SLO} = 1,$$

which is called a *side equation* or *sine equation.*

The side equation,

$$\frac{\sin SLN}{\sin SNL} \frac{\sin SOL}{\sin SLO} \frac{\sin SNO}{\sin SON} = 1,$$

gives the identical relation,

$$\frac{\text{side } SN}{\text{side } SL} \frac{\text{side } SL}{\text{side } SO} \frac{\text{side } SO}{\text{side } SN} = 1,$$

Hence, in forming a side equation we may proceed mechanically in this way. Write down the scheme

$$\frac{SN}{SL} \frac{SL}{SO} \frac{SO}{SN} = 1,$$

the numerator and denominator each being formed by the lines radiating from the point S in order of azimuth, and the first denominator being the second numerator. The side equation results from replacing the sides by the sines of the angles opposite to them.

The point S is called the *pole* of the quadrilateral for this equation.

It should be noted that side equations formed from a pole as starting-point are the most convenient to be used of the many that are possible. Not necessarily do all of the lines radiate from a point or pole that enter a side equation.

The side equations thus formed by the use of lines radiating from a selected pole are not the only ones possible. They are the *convenient* ones. As an example of a side equation not formed with a pole, take the following from Fig. 23, supposing L_1, L_2, L_3, to form one point L. The equation is

$$\frac{\sin ONS}{\sin SON}\frac{\sin NLS}{\sin LSN}\frac{\sin NOL}{\sin LNO}\frac{\sin LSO}{\sin OLS}=1,$$

This equation expressed in the form of ratios of sides is

$$\frac{SO}{SN}\frac{SN}{NL}\frac{NL}{OL}\frac{OL}{SO}=1.$$

It is evident that the sides involved do not ıl radiate from one point. Such side equations as these need not be used, as a sufficient number of the more conveniently formed side equations of the kind which involve a pole may always be secured.

Ex. — In the figure $ABCD_3D_1A$, the three angle equations,

$$\begin{aligned} D_1AB + ABD_1 + BD_1A &= 180^\circ + \epsilon_1, \\ ABC + BCA + CAB &= 180^\circ + \epsilon_2, \\ BCD_3 + CD_3B + D_3BC &= 180^\circ + \epsilon_3, \end{aligned}$$

given by the triangles D_1AB, ABC, BCD_3, may be satisfied, and yet the figure not be a perfect quadrilateral. Show by equating the values of BD_1 and BD_3 that the further condition necessary is

$$\frac{\sin ABD}{\sin BDA}\frac{\sin BCA}{\sin CAB}\frac{\sin CDB}{\sin BCD}=1.$$

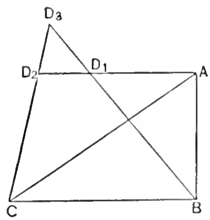

Fig. 16.

146. Position of Pole. — It is easily seen that in forming the side equation any vertex may be taken as pole. For plotting the figure from the angles of the triangles ONS, OLS, the side equation with pole at S means that the points L_1 and L_3 must coincide. The side equation with pole at N means that L_1, L_2 coincide, and with pole at O that L_2, L_3 coincide. If any one of these conditions is satisfied, the others are also satisfied, as each amounts to the same condition that L is not three points, but one point.

Similar reasoning will show that by plotting the figure from $LONS$, ONS, the side equations formed by taking the poles at N, L, S, mean that O is not three points but one point, and so on. Hence the side equation formed from any vertex as pole in connection with the angle equations fixes each point of the figure definitely and removes all contradictions from it.

It will be noticed that the reasoning is in no way affected by the line NL being sighted over in only one direction.

Ex. 1.— In a quadrilateral $ABCD$, in which all of the 8 angles are measured, show that of the 15 side equations that may be formed, not all of which are of the polar kind, 7 only are different in form, and that by taking the angle equations into account, all of them may be reduced to a single form.

Also show that there are 56 ways of expressing the 3 angle and 1 side equations necessary to determine the quadrilateral.

Ex. 2.— Examine the truth of the following statement. In a quadrilateral an angle equation may be replaced by a side equation, so that the quadrilateral may be determined by 3 angle equations and 1 side equation, 2 angle equations and 2 side equations, one angle equation and 3 side equations, the number of conditions remaining four, and the four not being all of one kind.

If the triangulation net, instead of involving quadrilaterals only, involves central polygons, such that, in computing the lengths of the sides, we can pass from one side to any other through a chain of triangles, the same process is followed in forming the side equations as in a quadrilateral.

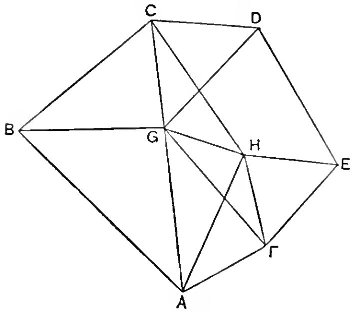

Fig 17.

Thus, in the figure which represents part of the triangulation of Lake Erie west of Buffalo Base, there are side equations from

The quadrilaterals $CDHG$, $GHFA$,
The pentagons $GABCH$, $HGDEF$.

The scheme for the pentagonal side equation $GABCH$, for example, would be just as in the case of a quadrilateral, taking G as pole,

$$\frac{GA}{GB}\,\frac{GB}{GC}\,\frac{GC}{GH}\,\frac{GH}{GA} = 1,$$

and the side equation,

$$\frac{\sin GBA}{\sin GAB}\,\frac{\sin GCB}{\sin GBC}\,\frac{\sin GHC}{\sin GCH}\,\frac{\sin GAH}{\sin GHA} = 1.$$

147. Reduction to the Linear Form. — Thus far we have considered the side equations in their rigorous form. But in order to carry through the solution by combining them with the other condition equations, they must be reduced to the linear form. We proceed to show how this may be done.

Let the side equation be

$$\frac{\sin V_1}{\sin V_2}\,\frac{\sin V_3}{\sin V_4}\cdots = 1, \tag{1}$$

where V_1, V_2, . . . denote the most probable values of the angles. Let M_1, M_2, . . . denote the measured values, and v_1, v_2, . . . the most probable corrections to these values; then the equation may be written,

$$\frac{\sin (M_1 + v_1)}{\sin (M_2 + v_2)}\,\frac{\sin (M_3 + v_3)}{\sin (M_4 + v_4)}\cdots = 1. \tag{2}$$

Taking the log of each side of this equation, and expanding by Taylor's theorem, we have, retaining the first powers of the corrections only,

$$\log \sin M_1 + \frac{d}{dM_1}(\log \sin M_1)\, v_1$$
$$- \left\{\log \sin M_2 + \frac{d}{dM_2}(\log \sin M_2)\, v_2\right\} + \cdots = 0, \tag{3}$$

which may be written in two forms for computation:

First, if the corrections to the angles are expressed in seconds, we may put

$$\frac{d}{dM_1}(\log \sin M_1) = \delta',$$

where δ' is the tabular difference for $1''$ for the angle M_1 in a table of log sines. Then we have,

$$\delta' v_1 - \delta'' v_2 + \cdots + \log \sin M_1 - \log \sin M_2 + \cdots = 0;$$

that is,

$$[\delta v] = l, \tag{4}$$

where l is a known quantity.

Secondly, we may replace

$$\frac{d}{dM_1}(\log \sin M_1) \text{ by mod } \sin 1'' \cot M_1,$$

where mod denotes the modulus of the common system of logarithms. Eq. 3 may then be arranged,

$$\cot M_1 v_1 - \cot M_2 v_2 + \cdots$$
$$= \frac{1}{10^7 \text{ mod } \sin 1''}(\log \sin M_2 - \log \sin M_1 + \cdots), \qquad (5)$$

if the seventh place of decimals is chosen as the unit.

The first of these two forms is preferred in the computing work by the computers of the Coast and Geodetic Survey.

148. Check Computation. — The side equation deduced from spherical triangles must also follow from the corresponding plane triangles, the angles of each spherical triangle being transformed according to Legendre's theorem; that is, for example, we should obtain the same constant term l by reducing to the linear form the equation,

$$\frac{\sin SLN}{\sin SNL}\,\frac{\sin SOL}{\sin SLO}\,\frac{\sin SNO}{\sin SON} = 1;$$

or the equation,

$$\frac{\sin\left(SLN - \frac{\epsilon_1}{3}\right)}{\sin\left(SNL - \frac{\epsilon_1}{3}\right)}\,\frac{\sin\left(SOL - \frac{\epsilon_2}{3}\right)}{\sin\left(SLO - \frac{\epsilon_2}{3}\right)}\,\frac{\sin\left(SNO - \frac{\epsilon_3}{3}\right)}{\sin\left(SON - \frac{\epsilon_3}{3}\right)} = 1,$$

where ϵ_1, ϵ_2, ϵ_3, denote the spherical excesses of the triangles SNL, SOL, and SON, respectively.

It is, in general, simpler to use the spherical angles than the plane angles. It affords a check on the accuracy of the numerical work to compute the side equation with both the spherical and the plane angles. It is hardly worth while, however, to spend the time required for this check, as it will take as long to apply the check as to have a duplicate computation made.

Ex. — The quadrilateral N. Base, S. Base, Oneota, Lester (Fig. 19). Take the pole at Lester.

We have the scheme $\qquad \dfrac{LS}{LN}\ \dfrac{LN}{LO}\ \dfrac{LO}{LS} = 1,$

from which we write down the side equation,

$$\frac{\sin LNS}{\sin LSN}\ \frac{\sin LON}{\sin LNO}\ \frac{\sin LSO}{\sin LOS} = 1\,;$$

that is, $\quad \dfrac{\sin(M_2 + v_2)}{\sin(M_5 + v_5)}\ \dfrac{\sin(M_8 + v_8)}{\sin(M_1 + v_1)}\ \dfrac{\sin(M_6 + v_6)}{\sin(M_7 + M_8 + v_7 + v_8)} = 1$

First Form of Reduction.

$$\begin{aligned}
\log\sin(113°\ 39'\ 05.07'' + v_2) &= 9.9618970 - \ \ 9.2\,v_2\\
\log\sin(\ 43°\ 46'\ 26.40'' + v_8) &= 9.8399903 + \ \ 2.0\,v_8\\
\log\sin(\ 70°\ 39'\ 24.60'' + v_6) &= 9.9747\underline{657} + \ \ 7.4\,v_6\\
&\qquad\quad\ 530\\
\log\sin(\ 47°\ 31'\ 20.41'' + v_5) &= 9.8677860 + 19.3\,v_5\\
\log\sin(124°\ 09'\ 40.69'' + v_1) &= 9.9177470 - 14.3\,v_1\\
\log\sin(\ 78°\ 27'\ 06.06'' + v_7 + v_8) &= 9.9911\underline{180} + \ \ 4.3\,(v_7 + v_8)\\
&\qquad\quad\ 510
\end{aligned}$$

Hence the side equation in the linear form is

$$14.3\,v_1 - 9.2\,v_2 - 19.3\,v_5 + 7.4\,v_6 - 4.3\,v_7 + 17.7\,v_8 + 20 = 0,$$

the unit being the seventh place of decimals.

CHECK off the constant term by computing the log sines after deducting from each angle $\frac{1}{3}$ of the spherical excess of the triangle to which it belongs.

ANGLE.	LOG SIN.	ANGLE.	LOG SIN.
113° 39′ 05.00″	9.9618970	47° 31′ 20.34″	9.8677858
43° 46′ 26.36″	9.8399903	124° 09′ 40.65″	9.9177471
70° 39′ 24.48″	99747656	78° 27′ 05.02″	9.9911180
	529		509
		+ 20	

agreeing closely with the value found from the spherical angles.

Second Form of Reduction.

LOG SIN.	LOG SIN.
$9.9618970 - 0.438\,v_2$	$9.8677860 + 0.916\,v_5$
$9.8399903 + 1.044\,v_8$	$9.9177470 - 0.679\,v_1$
$9.9747\underline{659} + 0.351\,v$	$9.9911180 + 0.204\,(v_7 + v_3)$

$$\begin{aligned}
&530\\
&\underline{510}\\
&\ \ 20 \quad \log 1.30103\\
\frac{1}{10^7 \bmod \sin 1''} &\quad \log \underline{8.67664}\\
&\qquad\ \ 9.97767 \qquad 0.955
\end{aligned}$$

and the side equation is

$$0.679\,v_1 - 0.438\,v_2 - 0.916\,v_5 + 0.351\,v_6 - 0.204\,v_7 + 0.840\,v_8 + 0.95 = 0.$$

This result may be checked in the same way as in the first form.

In reducing a side equation to the linear form, the coefficients of the corrections should be carried out to one place of decimals farther than the

absolute term. This for a short computation would be unnecessary, but in the reduction of an extensive triangulation net it is rendered necessary by the accumulation of errors from the dropping of the last figures in products and quotients.

It will be noticed that in the preceding example logarithmic sines have been carried to seven decimal places only. This is sufficient for the most accurate primary triangulation. An error of one in the seventh place of decimals corresponds to an error of less than 1 part in 4,000,000. In the most accurate primary triangulation, discrepancies of more than ten times the amount occur in at least half of the figures. The uniform present practice of the Coast and Geodetic Survey is to use 7-place logarithms. In the past, eight or more places have been frequently used.

149. We have seen that the coefficients of the corrections in a side equation are given by the differences for 1″ of the log sines of the angles, or by the cotangents of the angles that enter. There will be less liability to mistakes on account of misplaced decimal points, and less difficulty arising from omitted decimal places in the solution, and especially in connection with a check column, if the coefficients throughout the condition equations are of the same order of magnitude. Since the coefficients of the corrections in the angle equations are $+1$ or -1, it follows that it would be most convenient to put the side equations on the same footing as the angle equations. To do this we may divide the side equation by such a number as will make the average value of the coefficients equal to unity. This, for angles ordinarily met with in triangulation, would be effected by taking the sixth place of decimals as the unit in the side equation. Thus in our example, dividing by 10, which is approximately the mean of the coefficients, and which amounts to the same thing as expressing the log differences in units of the sixth place of decimals, the equation may be written

$$1.43\, v_1 - 0.92\, v_2 - 1.93\, v_5 + 0.74\, v_6 - 0.43\, v_7 + 1.77\, v_8 + 2.00 = 0.$$

It would have been equally correct to multiply each of the angle equations by 10, and so have put them on the same

footing as the side equations. Dividing the side equations is, however, to be preferred, as the coefficients are made smaller throughout, and the formation and solution of the normal equations are consequently easier.

A striking difference between condition equations and observation equations is here brought out. As a condition equation expresses a rigorous relation among the observed quantities altogether independent of observation, it may be multiplied or divided by any number without affecting that relation; with an observation equation, on the other hand, the effect would be to increase or diminish its weight. (Compare Art. 48.)

150. Position of Pole. — In a quadrilateral, taking any of the vertices as pole, the conclusion was reached in Art. 145 that any one of the resulting forms of side equation was as good as any other in satisfying the conditions imposed. But when a side equation is reduced to the linear form and is no longer rigorous, the question deserves further notice.

Two points are to be considered — precision of results and ease of computation. As regards the first, since the differences in a table of log sines are more sharply defined for small angles, and these differences are the coefficients of the unknowns in the side equation, it follows that in general that vertex should be chosen which allows the introduction of the acutest angles into the side equation.

Labor of computation will be saved by choosing the pole so that as few sine terms as possible enter. Thus by choosing the pole at O, the intersection of the diagonals (Fig. 22), the side equation would contain 8 terms, whereas, if taken at any of the vertices, only 6 terms would enter. Also, other things being equal, we should choose that pole which introduces the smallest number of unknowns into the equation, for then the normal equations would be more easily formed.

If the approximate form of solution in Art. 131 is employed, it is advantageous to choose the pole at the intersection, O, of the diagonals, as will be seen in the sequel.

151. Number of Side Equations in a Net. — A line being taken as a base, its extremities are known. To fix a third point, we must know the other two sides of the triangle of which this point is to be the vertex. Hence if we have a net of triangles connecting s stations, two of the stations being the ends of the base, we must have, in order to plot the figure, $2(s-2)$ lines besides the base; that is, $2s-3$ lines in all.

Starting from the base, each line in this figure can be computed in but one way, but any additional line, whether observed over in one or both directions, can be computed in two ways, and therefore gives rise to a side equation. If, then, the total number of lines in the figure is l, the number of side equations, as indicated by the number of superfluous lines, is

$$l - 2s + 3.$$

152. Check of the Total Number of Conditions. — The notation already given may be summarized as follows: In any figure or series of figures, l is the total number of lines, l_1 the number of lines observed over in one direction only, s the total number of stations, s_u the number of unoccupied stations, and n the number of angles measured which are independent in so far as local conditions are concerned.

At each station occupied, the number of locally independent angles is one less than the number of lines observed from that station, hence

$$n = 2l - l_1 - s + s_u. \tag{1}$$

From Art. 144 the number of angle equations in the figure is

$$l - l_1 - s + s_u + l. \tag{2}$$

From Art. 151 the number of side equations in the net is

$$l - 2s + 3. \tag{3}$$

More accurately, this is the number of side equations which are independent of each other and of the angle equations.

Adding (2) and (3), the total number of angle and side equations in the net is

$$2l - l_1 - 3s + s_u + 4.$$

Combining these with (1), it becomes $n - 2s + 4$, as was proved in Art. 141 to be the total number of conditions in a figure developed from a single base.

The condition equations referring to lengths, azimuth, latitude and longitude, which arise when there is more than one line in a figure fixed by previous adjustment, will be treated later.

153. Manner of Selecting the Angle and Side Condition Equations. — In the selection of side and angle equations in a triangulation net, four dangers must be guarded against. First, that some necessary condition equation may be omitted; second, that some unnecessary condition equation may be introduced; third, that a condition equation which is chosen may not be independent of those already selected; and fourth, that the condition equations used may be so selected from the many available that the solution of the normal equations will be an unstable one; that is, a solution in which the effect of omitted decimal places on the derived values of the required unknowns is large, and in which it is therefore necessary to carry a large number of decimal places in the solution to secure the unknowns with certainty to a small number of decimal places.

A good method of avoiding the first, second, and third of these dangers is to start from some line as base and plot the figure point by point. As each point is added, draw all observed lines connecting it with points previously located, and express the conditions arising from the new lines. For each new point, the number of new angle equations is one less than the number of lines observed in both directions connecting it with previous points, and the number of new side equations is two less than the number of lines connecting it with previous points, regardless of whether these lines are observed in both directions or only in one direction.

For example, let Fig. 18 represent a triangulation net, plotted in detail as follows: First draw the line connecting the points Tobacco Row and Spear. Add the new point Long, and con-

nect with Tobacco Row and Spear. This furnishes one angle equation from the triangle Long–Tobacco Row–Spear. Next add the new point Smith, and draw lines from it to Tobacco Row, Spear, and Long. This furnishes two new angle equations corresponding to the two new triangles, and one side equation. Complete the figure by adding the new point Flat Top, and draw four lines to Tobacco Row, Spear, Long, and Smith. This introduces three new angle equations and two new side equations, making the total number of angle equations six, and of side equations three. These numbers may be checked by the formulas in sec. 154.

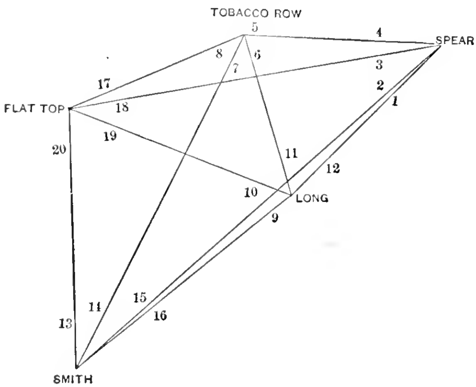

Fig. 18.

If in this triangulation net the line Spear–Smith had been observed in but one direction, Spear to Smith, and Flat Top had been an unoccupied station, the drawing of the figure in the same three steps would have indicated respectively one angle equation, one angle equation and one side equation, and two side equations. The total number of side equations would have been three as before, and of angle equations two. These numbers may be checked by the formulas in sec. 154.

154. If the above process of selecting the condition equations is followed strictly, there will be little danger of choosing mutually dependent condition equations. If, however, such mutually dependent equations have been chosen, it will become evident in the course of the solution by the appearance there of two equations which are identical. In this case one of the correlates becomes indeterminate. The danger of selecting such condition equations that the solution will be somewhat unstable

is a much more difficult one to avoid. In such cases the skill of the expert computer gives him a decided advantage.

The computer may be guided by the following suggestions and conditions based on experience. These suggestions will tend to make the solutions of the equations less laborious as well as more stable.

1. In selecting angle equations, preference should be given to the triangles which have one or two sides on the exterior of the figure. This tends to avoid entanglements with other conditions.

It is expedient to exclude triangles with small angles in order to avoid entanglement with side equations having large coefficients. It is often desirable, though of less importance, to exclude triangles that adjoin small angles, and so have a side in common with them.

2. In selecting the side equations, it is desirable, as already indicated in Art. 150, to secure large coefficients, and therefore small angles should be used. If, however, such side equations are selected that the small angles of the figure are used more than once, it may be found that the solution is unstable because in some of the normal equations there are side coefficients of about the same magnitude as the diagonal coefficient. The rule in selecting side equations should therefore be to use the small angles of the figure once and only once.

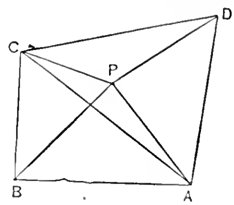

Fig. 19.

3. While it is true in general that time will be saved by using side equations having a small number of terms, there are exceptions to the rule. For example, in dealing with the following figure, experience shows that it is advisable to use a side equation having its pole at P and involving all four of the lines which radiate from it, and containing eight terms, although a sufficient number of side equations could be written, each of which would contain but six terms. Experience shows that in such a case as this the solution may be somewhat unstable,

unless this side equation of large scope is used, apparently because the points B and D, which are not connected by a line of sight, are not otherwise sufficiently bound together.

4. In the process of elimination, it is advisable to avoid the introduction into an explicit function of an unknown quantity from which the corresponding original equation is free.

In line 4, Table A, the value of x contains the unknown quantity y which was absent from the second equation. Had the third equation changed places with the first, the terms of the original second equation might have been introduced directly into Table A, and the number of lines in Table B would have been two less, and the number of columns one less.

The order of solution in a figure adjustment can be best decided by inspection of the figure. The work should commence with an angle equation from a triangle having a side (or better, two sides) on the exterior of the figure; and no angle equation from a triangle with a new interior side should ever be introduced till after the entrance of every angle equation not thus exposed to entanglement with conditions yet untouched. A side equation should usually be postponed till after the introduction of all the angle equations that relate to the same points and no others, but should immediately follow them, so as to precede all equations that extend beyond its domain into new territory.

The suggestions in the preceding paragraphs will be illustrated in Art. 180, in connection with the method of directions.

155. Adjustment of the Quadrilateral *NSOL* (Fig. 10).

The method of forming the condition equations having now been explained, we are ready to adjust the quadrilateral *NSOL*, as promised in Art. 137.

The condition equations have all been formed in the preceding sections. Collecting them, we have:

Local equations (Ex. 1, 2, Art. 139),

$$v_1 + v_2 + v_3 = -1.37,$$
$$v_4 + v_5 - v_6 = -1.07.$$

Angle equations (Ex. Art. 143),

$$v_3 + v_4 + v_7 = -0.48,$$
$$v_6 + v_7 + v_8 + v_9 = -1.10.$$

Side equation, the unit being the sixth place of decimals (Ex. Art. 148),

$$1.43\,v'_1 - 0.92\,v'_2 - 1.93\,v_5 + 0.74\,v_6 - 0.43\,v'_7 + 1.77\,v'_8 = -2.00.$$

The methods of solution have been explained in Chapter V, and we shall proceed in the order there given for the three forms.

156. *First Solution — Method of Independent Unknowns*

There being 9 unknowns and 5 condition equations connecting them, there must be 4 independent unknowns. We shall choose v'_1, v'_2, v'_4, v'_5. Expressing all of the unknowns in terms of these four, we write the equations in the form of observation equations, as follows (see Art. 118):

$$\begin{array}{lllllll}
v'_1 = + & v_1 & & & & \text{weight} & 2 \\
v'_2 = & + & v_2 & & & " & 2 \\
v_3 = - & v_1 - & v_2 & & - 1.37 & " & 14 \\
v'_4 = & & + & v_4 & & " & 23 \\
v_5 = & & & + & v'_5 & " & 6 \\
v'_6 = & & + & v_4 + & v_5 + 1.07 & " & 7 \\
v_7 = + & v_1 + & v_2 - & v_4 & + 0.89 & " & 31 \\
v'_8 = - 0.565\,v'_1 & + 0.763\,v_2 & - 0.661\,v_4 & + 0.672\,v'_5 & - 1.361 & " & 1 \\
v_9 = - 0.435\,v'_1 & - 1.763\,v_2 & + 0.661\,v_4 & - 1.672\,v_5 & - 1.699 & " & 8
\end{array} \qquad (1)$$

Hence the normal equations

v'_1	v'_2	v'_4	v'_5	CONST.	
+ 48.83	+ 50.70	− 32.93	+ 5.44	− 53.45	. . . .
. . . .	+ 72.45	− 40.83	+ 24.09	− 69.70	. . . .
. . . .	. . . .	+ 64.93	− 2.29	+ 28.18	. . . .
. . . .	. . . .	. . . .	+ 35.82	− 29.30	. . . .
. . . .	. . . .	. . . .	. . . .	+ 83.79	= [pll]

Solving these equations (Art. 157), we have the values of the corrections,

$$v'_1 = -0.82'', \qquad v'_4 = -0.22'',$$
$$v'_2 = -0.36'', \qquad v'_5 = -0.47'',$$

and thence from the condition equations,

$$v'_3 = -0.19'', \qquad v'_8 = -1.33'',$$
$$v'_6 = +0.38'', \qquad v'_9 = -0.08''.$$
$$v_7 = -0.07'',$$

These corrections applied to the measured values of the angles give the most probable values as follows:

$$M_1 = 124^\circ\ 09'\ 39.87'', \qquad M_6 = 70^\circ\ 39'\ 24.98'',$$
$$M_2 = 113^\circ\ 39'\ 04.71'', \qquad M_7 = 34^\circ\ 40'\ 39.59'',$$
$$M_3 = 122^\circ\ 11'\ 15.42'', \qquad M_8 = 43^\circ\ 46'\ 25.07'',$$
$$M_4 = 23^\circ\ 08'\ 05.04'', \qquad M_9 = 30^\circ\ 53'\ 30.73''.$$
$$M_5 = 47^\circ\ 31'\ 19.94'',$$

157. The Precision of the Adjusted Values.

(*a*) To find the m. s. e. of an observation of the unit of weight (Arts. 105, 108).

From the above values of the residuals v,

$$[pv^2] = 7.53.$$

Check of $[pv^2]$. Carrying through the solution of the normal equations the extra column required by the sum $[pll]$, we find (p. 209),

$$[pv^2] = 7.54.$$

Hence,

$$\mu = \sqrt{\frac{7.54}{9-4}}$$
$$= \pm 1.23''.$$

(*b*) To find the weight and m. s. e. of the adjusted value of an angle.

Take the angle NLS. Proceeding as in Art. 108, we have,

$$F = NLS$$
$$= 180 + \epsilon - (M_2 + v_2 + M_5 + v_5).$$
$$\therefore\ dF = -v_2 - v_5.$$

Hence, from the extra column, the sixth, carried through the solution of the normal equations (p. 209),

$$u_F = 0.053,$$

and therefore,

$$\mu_F = 1.23\sqrt{0.053}$$
$$= 0.28''.$$

(*c*) To find the weight and m. s. e. of the adjusted value of a side, the base, NS, being supposed to be free from error.

Let us take the side OL. We have,

$$F = OL$$
$$= NS \frac{\sin ONS}{\sin SON} \frac{\sin LSO}{\sin OLS}$$
$$= NS \frac{\sin (M_3 + v_3)}{\sin (M_7 + v_7)} \frac{\sin (M_6 + v_6)}{\sin (M_9 + v_9)}.$$

For check we shall proceed in two ways.

(1) Expand F directly; then,

$$dF = \left(\frac{\delta F}{\delta M_3} v_3 + \frac{\delta F}{\delta M_6} v_6 + \frac{\delta F}{\delta M_7} v_7 + \frac{\delta F}{\delta M_9} v_9\right) \sin 1''$$
$$= -0.0505\, v_3 + 0.0282\, v_6 - 0.1160\ {}_7 - 0.1342\, v_9$$
$$= -0.007\ v_1 + 0.171\ v_2 + 0.056\ {}_4 + 0.253\ v_5,$$

by substituting for v_3, v_6, v_7, v_9, their values from equations (1).

Carry through the solution of the normal equations the extra column required by these coefficients, and

$$u_F = 0.0019.$$

Hence,

$$\mu_F = 1.23\sqrt{0.0019}$$
$$= 0.05\ m.$$

(2) Take logs of both members of the equation ; then,

$$\log F = \log NS + \log \sin (M_3 + v_3) + \log \sin (M_6 + v_6)$$
$$- \log \sin (M_7 + v_7) - \log \sin (M_9 + v_9).$$

But since NS is constant, we have, in units of the sixth place of decimals,

$$d \log F = - 1.33\, v_3 + 0.74\, v_6 - 3.04\, v_7 - 3.52\, v_9$$
$$= - 0.18\, v_1 + 4.50\, v_2 + 1.45\, v_4 + 6.63\, v_5, \text{ from equations (1).}$$

Hence, from the last column added to the solution of the normal equations,

$$u_{\log F} = 1.50 \text{ in units of the sixth place of decimals.}$$

Also, $$\mu_{\log F} = 1.23 \sqrt{1.50}$$
$$= 1.5 \text{ in units of the sixth place of decimals.}$$

Now, since $d \log F = \dfrac{dF}{F} \text{ mod.}$

and $F = 16556\, m,$

$\therefore\ \mu F = 0.06\, m.$

The solution of the normal equations, with the extra columns required by the weight determinations, is as follows :

v_1	v_2	v_4	v_5	l	f (angle).	f (side).	f (side).
+ 48.83	+ 50.70	− 32.93	+ 5.44	− 53.45		− 0.007	− 0.18
	+ 72.45	− 40.83	+ 24.09	− 69.70	− 1	+ 0.171	+ 4.50
		+ 64.93	− 2.29	+ 28.18		+ 0.056	+ 1.45
			+ 35.82	− 29.30	− 1	+ 0.253	+ 6.63
			[pll] =	+ 83.79			
+ 1	+ 1.038	− 0.674	+ 0.1114	− 1.0946		− 0.000	− 0.004
	+ 19.808	− 6.643	+ 18.4420	+ 14.2038	− 1	+ 0.176	+ 4.687
		+ 42.725	+ 1.3784	+ 7.8652		+ 0.053	+ 1.328
			+ 35.2140	+ 23.3454	− 1	+ 0.253	+ 6.650
				+ 25.2830	0	0	+ 0.001
	+ 1	− 0.335	+ 0.931	+ 0.7176	+ 0.050	+ 0.009	+ 0.237
		+ 40.497	+ 7.563	+ 12.6322	− 0.335	+ 0.112	+ 2.900
			+ 18.044	+ 10.1114	− 0.069	+ 0.080	+ 2.287
				+ 15.0910	+ 0.050	+ 0.002	+ 1.109
		+ 1	+ 0.187	+ 0.3112	− 0.008	+ 0.003	+ 0.072
			+ 16.632	+ 7.753	− 0.006	+ 0.069	+ 1.745
				+ 11.151	+ 0.003	+ 0.000	+ 0.208
			+ 1	+ 0.466	− 0.000	+ 0.000	+ 0.105
			[pvv.] =	+ 7.538	0	0	+ 0.183
					0.000	0.000	0.000
					0.050	0.002	1.109
					0.003	0.000	0.208
					.000	0.000	0.183
					0.0533	0.002	1.500
					$= u_F$	$= u_F$	$= u_F$

The solution has been carried to four places of decimals in certain parts, on account of loss of accuracy arising from dropping figures in multiplications. The resulting values of the corrections have been cut down to two places of decimals. The work was done with a machine, as explained on p. 106, the reciprocals of the diagonal terms being used so as to avoid divisions. Thus the first reciprocal is 0.02048.

158. *Second Solution — Method of Correlates.*

Arranging the condition equations in tabular form, we have

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	
weights 2	2	14	23	6	7	31	1	8	
+ 1.43	− 0.92	. . .	. . .	− 1.93	+ 0.74	− 0.43	+ 1.77	. . .	− 2.00
+ 1	+ 1	+ 1	. . .	. . .	. . .	. . .	. . .	. . .	− 1.37
. . .	. . .	+ 1	+ 1	. . .	. . .	+ 1	. . .	. . .	− 0.48
. . .	. . .	. . .	+ 1	+ 1	− 1	. . .	. . .	. . .	− 1.07
. . .	. . .	. . .	. . .	. . .	+ 1	− 1	+ 1	+ 1	− 1.10

The Correlate Equations.

	I.	II.	III.	IV.	V.
$2\,v_1 =$	+ 1.43	+ 1	. . .	. . .	. . .
$2\,v_2 =$	− 0.92	+ 1	. . .	. . .	. . .
$14\,v_3 =$	. . . .	+ 1	+ 1	. . .	. . .
$23\,v_4 =$	. . . .	. . .	+ 1	+ 1	. . .
$6\,v_5 =$	− 1.93	. . .	. . .	+ 1	. . .
$7\,v_6 =$	+ 0.74	. . .	. . .	− 1	+ 1
$31\,v_7 =$	− 0.43	. . .	+ 1	. . .	+ 1
$v_8 =$	+ 1.77	. . .	. . .	. . .	+ 1
$8\,v_9 =$	. . . .	. . .	. . .	. . .	+ 1

The Normal Equations.

I.	II.	III.	IV.	V.	l.
+ 5.284	+ 0.255	− 0.014	− 0.427	+ 1.862	− 2.00
. . . . .	+ 1.071	+ 0.071	. . . . .	. . . . .	− 1.37
. . . . .	. . . . .	+ 0.147	+ 0.043	+ 0.032	− 0.48
. . . . .	. . . . .	. . . . .	+ 0.353	− 0.143	− 1.07
. . . . .	. . . . .	. . . . .	. . . . .	+ 1.300	− 1.10

The solution of these equations gives (see page 212)

$$
\begin{aligned}
\text{I.} &= -\ 0.3973, \\
\text{II.} &= -\ 1.0749, \\
\text{III.} &= -\ 1.6006, \\
\text{IV.} &= -\ 3.5721, \\
\text{V.} &= -\ 0.6301.
\end{aligned}
$$

Substituting these values in the correlate equations, the same values of the corrections result as before. Also,

$$[pv^2] = 7.53.$$

159. The Precision.

(*a*) To find the m. s. e. μ of an observation of weight unity.

From the values of v we find directly,

$$[pv^2] = 7.53.$$

Checks of $[pv^2]$. These are worked out in the solution of the normal equations on p. 178, according to the formulas of Art. 121, and give 7.54 and 7.55 respectively.

Hence, taking the mean, $[pv^2] = 7.54$, and the number of conditions being 5,

$$
\begin{aligned}
\mu &= \sqrt{\frac{7.54}{5}} \\
&= 1.23'', \text{ as before.}
\end{aligned}
$$

Compare Ex. 2, Art. 122.

(*b*) To find the weight and m. s. e. of the adjusted value of an angle.

Take the angle *NLS*.

$$\therefore\ dF = -\ v_2 - v_5.$$

From the values of u, a, b, . . . in the condition equations in connection with the values of f given by this function, we have

$$
\begin{aligned}
[uaf] &= +0.782, & [udf] &= -\ 0.167, \\
[ubf] &= -0.500, & [uef] &= \quad 0. \\
[ucf] &= \quad 0, & [uff] &= +\ 0.667.
\end{aligned}
$$

Hence, from the seventh column in the solution of the normal equations below,

$$u_F = 0.053$$

and

$$
\begin{aligned}
\mu_F &= 1.23\sqrt{0.053} \\
&= 0.28''.
\end{aligned}
$$

Compare Ex. 4, Art. 125.

(*c*) To find the weight and mean-square error of the adjusted value of a side, the base being free from error.

Take the side Oneota–Lester.

As in (*c*), Art. 157, we have,

$$dF = -\ 0.0505\,v_3 + 0.0282\,v_6 - 0.1160\,v_7 - 0.13425\,v_9.$$

Also from the condition equations,

$$[uaf] = + 0.0046, \qquad [udf] = - 0.0040,$$
$$[ubf] = - 0.0036, \qquad [uef] = - 0.0165,$$
$$[ucf] = - 0.0073, \qquad [uff] = + 0.0030.$$

Hence, from the eighth column in the solution of the normal equations,

$$u_F = 0.0023,$$

and finally,

$$\mu_F = 1.23 \sqrt{.0023}$$
$$= 0.06\ m.$$

Solution of the Normal Equations.

I.	II.	III.	IV.	V.	l	f (Angle)	f (Side).
+ 5.284	+ 0.255 + 1.071	− 0.014 + 0.071 + 0.147	− 0.427 + 0.043 + 0.353	+ 1.862 + 0.032 − 0.143 + 1.300	− 2.00 − 1.37 − 0.48 − 1.07 − 1.10	+ 0.782 − 0.500 − 0.167 + 0.667	+ 0.0046 − 0.0036 − 0.0073 − 0.0040 − 0.0165 + 0.0030
+ 1	+ 0.0483 + 1.0587	− 0.0026 + 0.0717 + 0.1470	− 0.0808 + 0.0206 + 0.0419 + 0.3185	+ 0.3524 − 0.0902 + 0.0369 + 0.0075 + 0.6438	− 0.3785 − 1.2731 − 0.4853 − 1.2316 − 0.3952 + 0.7570	+ 0.1480 − 0.5377 + 0.0021 − 0.1035 − 0.2756 + 0.5510	+ 0.0009 − 0.0038 − 0.0073 − 0.0036 − 0.0182
	+ 1	+ 0.0677 + 0.1421	+ 0.0204 + 0.0404 + 0.3181	− 0.0851 + 0.0434 + 0.0093 + 0.6361	− 1.2025 − 0.3991 − 1.2068 − 0.5037 + 1.5309	− 0.5078 + 0.0385 − 0.0930 − 0.3214 + 0.2780	− 0.0036 − 0.0070 − 0.0035 − 0.0185 + 0.0630
		+ 1	+ 0.2843 + 0.3066	+ 0.3054 − 0.0030 + 0.6228	− 2.8086 − 1.0933 − 0.3818 + 1.1209	+ 0.2709 − 0.1039 − 0.3332 + 0.2676	− 0.0493 − 0.0015 − 0.0164 + 0:0027
Values of the Unknowns: I. = − 0.3973, II. = − 1.0749, III. = − 1.6006, IV. = − 3.5721, V. = − 0.6301.			+ 1	− 0.0098 + 0.6228	− 3.5659 − 0.3024 + 3.8986	− 0.3389 − 0.3342 + 0.2324	− 0.0049 − 0.0164 + 0.0027
				+ 1	− 0.6301 = V + 0.2472	− 0.5366 + 0.0531 $= u_F$	− 0.0264 + 0.0023 $= u_F$
I. × l' = − 0.3973 × − 2.00 = 0.79 II. × l'' = − 1.0749 × − 1.37 = 1.47 III. × l''' = − 1.6006 × − 0.48 = 0.77 IV. × l'''' = − 3.5721 × − 1.07 = 3.82 V. × l''''' = − 0.6301 × − 1.10 = 0.69 7.54 = [pvv]					0.7570 1.5309 1.1209 3.8986 0.2472 7.5546 = [pvv]		

The values of [pv^2] are found from Equations 2, 3, Art. 121.

160. *Third Solution — Solution in Two Groups.*

The form given in Art. 128 is followed.

THE LOCAL ADJUSTMENT.

(a) At North Base.

THE OBSERVATION EQUATIONS.

p	(x_1)	(x_2)	l
2	$+1$	...	0.
2	...	$+1$	0.
14	-1	-1	-1.37

THE NORMAL EQUATIONS.

$$\begin{aligned} &(x_1)\ \ (x_2) \\ &16 + 14 = -19.18 = [pal] \text{ suppose,} \\ &14 + 16 = -19.18 = [pbl] \text{ suppose.} \end{aligned}$$

Solving in general terms,

$$(x_1) = +0.267\,[pal] - 0.233\,[pbl],$$
$$(x_2) = -0.233\,[pal] + 0.267\,[pbl].$$

Hence,

$$(x_1) = -0.64'',$$
$$(x_2) = -0.64'',$$
$$(x_3) = +0.64'' + 0.64'' = 1.37'',$$
$$= -0.09'',$$

and

LOCAL ANGLES.

$$124^\circ\ 09'\ 40.05'',$$
$$113^\circ\ 39'\ 04.43'',$$
$$122^\circ\ 11'\ 15.52''.$$

To find the m. s. e. of a single observation.

The value of $[pv^2] = [pxx] = 1.75$.

Hence, for this station, the number of conditions being $3 - 2$,

$$\mu = \sqrt{\frac{1.75}{3-2}}$$
$$= 1.3''.$$

(b) At South Base.

THE OBSERVATION EQUATIONS.

p	(x_4)	(x_5)	l
23	$+1$		0.
6		$+1$	0.
7	$+1$	$+1$	-1.37

THE NORMAL EQUATIONS.

$$\begin{array}{l} (x_4) \quad (x_5) \\ 30 + 7 = -7.49, \\ 7 + 13 = -7.49. \end{array}$$

Hence

$$\begin{aligned} (x_4) &= -0.13'', \\ (x_5) &= -0.50'', \\ (x_6) &= -0.13'' - 0.50'' + 1.07'', \\ &= +0.44''. \end{aligned}$$

LOCAL ANGLES.

$$\begin{array}{lll} 23^\circ & 08' & 05.13'', \\ 47^\circ & 31' & 19.91'', \\ 70^\circ & 39' & 25.04''. \end{array}$$

Also,

$$[pv^2] = [pxx] = 3.24.$$

$$\therefore \quad \mu = \sqrt{\frac{3.24}{3-2}}$$

$$= 1.8''.$$

THE GENERAL ADJUSTMENT.

Most Probable Angles.

At N. Base,	124°	09′	40.05″ + (1),
	113°	39′	04.43″ + (2),
	122°	11′	15.52″ − (1) − (2).
At S. Base,	23°	08′	05.13″ + (4),
	47°	31′	19.91″ + (5),
	70°	39′	25.04″ + (4) + (5).
At Oneota,	34°	40′	39.66″ + (7),
	43°	46′	26.40″ + (8).
At Lester,	30°	53′	30.81″ + (9).

The Angle and Side Equations.

(*a*) Triangle, N. Base, S. Base, Oneota.

$$\begin{array}{lll} \text{Angle } SNO & 122^\circ\ 11'\ 15.52'' & - (1) - (2) \\ \quad\text{"}\quad NSO & 23^\circ\ 08'\ 05.13'' & + (4) \\ \quad\text{"}\quad NOS & 34^\circ\ 40'\ 39.66'' & + (7) \\ \hline \text{Sum} \quad = & 180^\circ\ 00'\ 00.31'' & \\ 180 + \epsilon = & 180^\circ\ 00'\ 00.05'' & \\ \hline \end{array}$$

$$0 = 0.26'' - (1) - (2) + (4) + (7)$$

(b) Triangle Lester, Oneota, S. Base.

$$
\begin{array}{lll}
\text{Angle } NSO & 70^\circ\ 39'\ 25.04'' + (4) + (5) \\
\quad " \quad SOL & 78^\circ\ 27'\ 06.06'' + (7) + (8) \\
\quad " \quad OLS & 30^\circ\ 53'\ 30.81'' + (9) \\
\hline
& 180^\circ\ 00'\ 01.91'' \\
& 180^\circ\ 00'\ 00.37'' \\
\hline
\end{array}
$$

$$0 = 1.54'' + (4) + (5) + (7) + (8) + (9)$$

(c) Quadrilateral N. Base, S. Base, Oneota, Lester.

$$\frac{\sin LNS}{\sin LNO}\ \frac{\sin LSO}{\sin NSL}\ \frac{\sin LON}{\sin LOS} = 1.$$

$$
\begin{array}{ll}
LNS = 113^\circ\ 39'\ 04.43'' + (2), & LNO = 124^\circ\ 09'\ 40.05'' + (1), \\
LSO = \ \ 70^\circ\ 39'\ 25.04'' + (4) + (5), & NSL = \ \ 47^\circ\ 31'\ 19.91'' + (5), \\
LON = \ \ 43^\circ\ 46'\ 26.40'' + (8), & LOS = \ \ 78^\circ\ 27'\ 06.06'' + (7) + (8).
\end{array}
$$

$$
\begin{array}{ll}
9.9618975{,}6 - \ \ 9{,}22\ (2) & 9.9177479.3 - 14{,}29\ (1) \\
9.9747660{,}1 + \ \ 7{,}39\ \{(4) + (5)\} & 9.8677849{,}8 + 19{,}28\ (5) \\
9.8399903{,}4 + 21{,}98\ (8) & 9.9911180{,}3 + \ \ 4{,}30\ \{(7) + (8)\} \\
\hline
539{,}1 & 509{,}4 \\
509{,}4 & \\
\hline
29{,}7 &
\end{array}
$$

Check by deducting $\frac{1}{3}$ of the spherical excesses of the triangles from the angles.

$$
\begin{array}{ll}
113^\circ\ 39'\ 04.36'', & 124^\circ\ 09'\ 40.01'', \\
70^\circ\ 39'\ 24.92'', & 47^\circ\ 31'\ 19.84'', \\
43^\circ\ 46'\ 26.36'', & 78^\circ\ 27'\ 05.93''.
\end{array}
$$

$$
\begin{array}{rr}
9.9618976{,}2 & 9.9177479{,}9 \\
9.9747659{,}3 & 9.8677848{,}6 \\
9.8399902{,}5 & 9.9911179{,}8 \\
\hline
38{,}0 & 8{,}3 \\
8{,}3 & \\
\hline
29{,}7 &
\end{array}
$$

The two methods agree well.

A glance at the log differences for 1″ shows that by expressing them in units of the sixth place of decimals their average value is unity nearly. We have, then, for the side equation,

$$1.43\ (1) - 0.92\ (2) + 0.74\ (4) - 1.19\ (5) - 0.43\ (7) + 1.77\ (8) + 2.97 = 0.$$

The Weight Equations.

$$(1) = -0.233\,\boxed{1} + 0.267\,\boxed{2}$$

$$(2) = +0.267\,\boxed{1} - 0.233\,\boxed{2}$$

$$(4) = +0.038\,\boxed{4} - 0.021\,\boxed{5}$$

$$(5) = -0.021\,\boxed{4} + 0.088\,\boxed{5}$$

$$(7) = +0.032\,\boxed{7}$$

$$(8) = +1.000\,\boxed{8}$$

$$(9) = +0.125\,\boxed{9}$$

The Correlate Equations.

	I.	II.	III.	CHECK.
$\boxed{1}$ =	-1		$+1.43$	$= 0.43$
$\boxed{2}$ =	-1		-0.92	$+1.92$
$\boxed{4}$ =	$+1$	$+1$	$+0.74$	-2.74
$\boxed{5}$ =		$+1$	-1.19	$+0.19$
$\boxed{7}$ =	$+1$	$+1$	-0.43	-1.57
$\boxed{8}$ =		$+1$	$+1.77$	-2.77
$\boxed{9}$ =		$+1$		-1.00

The check is formed by adding each horizontal row (Art. 78).

Expression of the Corrections in Terms of the Correlates.

	I.	II.	III.	CHECK.
	$+0.233$		-0.333	$+0.100$
	-0.267		-0.246	$+0.513$
(1) =	-0.034		-0.579	$+0.613$
	-0.267		$+0.382$	-0.115
	$+0.233$		$+0.214$	-0.447
(2) =	-0.034		$+0.596$	-0.562
	$+0.038$	$+0.038$	$+0.028$	-0.104
		-0.021	$+0.024$	-0.004
(4) =	$+0.038$	$+0.017$	$+0.052$	-0.108
	-0.021	-0.021	-0.016	$+0.058$
		$+0.088$	-0.105	$+0.017$
(5) =	-0.021	$+0.067$	-0.121	$+0.075$
(7) =	$+0.032$	$+0.032$	-0.014	-0.050
(8) =		$+1.$	$+1.770$	-2.770
(9) =		$+0.125$		-0.125

The Corrections in Terms of the Correlates (Collected).

	I.	II.	III.
(1) =	− 0.034		− 0.579
(2) =	− 0.034		+ 0.596
(4) =	+ 0.038	+ 0.017	+ 0.052
(5) =	− 0.021	+ 0.067	− 0.121
(7) =	+ 0.032	+ 0.032	− 0.014
(8) =		+ 1.000	+ 1.770
(9) =		+ 0.125	

Formation of the Normal Equations.

	I.	II.	III.	Check.
− (1) =	+ 0.034		+ 0.579	− 0.613
− (2) =	+ 0.034		− 0.596	+ 0.562
+ (4) =	+ 0.038	+ 0.017	+ 0.052	− 0.108
+ (7) =	+ 0.032	+ 0.032	− 0.014	− 0.050
	+ 0.138	+ 0.049	+ 0.021	+ 0.209
(4) =		+ 0.017	+ 0.052	− 0.108
(5) =		+ 0.067	− 0.121	+ 0.075
(7) =		+ 0.032	− 0.014	− 0.050
(8) =		+ 1.	+ 1.770	− 2.770
(9) =		+ 0.125		− 0.125
	+ 0.049	+ 1.241	+ 1.687	− 2.978

	I.	II.	III.	Check.
+ 1.43(1)			+ 0.852	− 0.804
− 0.92(2)			+ 0.533	− 0.564
+ 0.74(4)			+ 0.038	− 0.080
− 1.19(5)			+ 0.144	− 0.089
− 0.43(7)			+ 0.006	+ 0.022
+ 1.77(8)			+ 3.133	− 4.903
	+ 0.021	+ 1.687	+ 4.706	− 6.418

The Normal Equations (Collected).

I.	II.	III.	
+ 0.138	+ 0.049	+ 0.021	= − 0.260
	+ 1.241	+ 1.687	= − 1.540
		+ 4.706	= − 2.970

The solution of these equations gives (page 220)

I. = − 1.597
II. = − 0.642
III. = − 0.394

Substitute for I., II., III., their values in (4), and we have the general corrections.

Adding the local corrections and general corrections together, the total corrections to the measured angles result and are as follows:

	LOCAL.	GENERAL.	TOTAL.	p	pv^2	FINAL ANGLES.		
	$''$	$''$	$''$			°	′	″
$x_1 =$	-0.64	-0.18	$= -0.82$	2	1.34	124	09	39.87
$x_2 =$	-0.64	$+0.28$	$= -0.36$	2	.26	113	39	4.71
$x_3 =$	-0.09	-0.10	$= -0.19$	14	.50	122	11	15.42
$x_4 =$	-0.13	-0.09	$= -0.22$	23	1.10	23	08	5.04
$x_5 =$	-0.50	$+0.04$	$= -0.46$	6	1.27	47	31	19.95
$x_6 =$	$+0.44$	-0.05	$= +0.39$	7	1.06	70	39	24.99
$x_7 =$	....	-0.07	$= -0.07$	31	.15	34	40	39.59
$x_8 =$	....	-1.33	$= -1.33$	1	1.77	43	46	25.07
$x_9 =$	....	-0.08	$= -0.08$	8	.05	30	53	30.73
				$[pv^2] = 7.50$				

Number of local conditions $= 2$
Number of general conditions $= 3$
Total $= 5$

The method of solution just given is substantially the same as that employed on the survey of the Great Lakes between Canada and the United States by the U. S. Engineers.

161. The Precision of the Adjusted Values.

(*a*) To find the m. s. e. of an observation of weight unity.

Computation of $[pv^2]$.

(1) From the preceding table $[pv^2]$ has been found directly; thus,

$$[pv^2] = 7.50.$$

(2) Check (Art. 129). From the station adjustments find $[v^\circ v^\circ]$.

N. Base gives (p. 213) 1.75
S. Base gives (p. 214) 3.24
4.99 $= [v^\circ v^\circ]$.

From the general adjustment find $[ww]$.

$$(\alpha)\quad l_0' \times \text{I.} = -0.26 \times -1.597 = +0.42$$
$$l_0'' \times \text{II.} = -1.54 \times -0.642 = +0.99$$
$$l_0''' \times \text{III.} = -2.97 \times -0.394 = +1.16$$
$$+2.57$$

$$(\beta)\quad l_0' \times \frac{l_0'}{[a\text{A}]} = -0.26 \times -1.885 = +0.49$$

$$l_0''.1 \times \frac{l_0''1}{[b\text{B}.1]} = -1.45 \times -1.183 = +1.72$$

$$l_0'''.2 \times \frac{l_0'''2}{[cC.2]} = -0.94 \times -0.394 = +0.37$$

$$+\overline{2.58}$$

$$\therefore [ww] = 2.58$$

and $$[pv^2] = 4.99 + 2.58$$
$$= 7.57.$$

Hence, taking the mean of the values of $[pv^2]$,

$$\mu = \sqrt{\frac{7.54}{2+3}} = 1.23'',$$

there being 2 local conditions and 3 net conditions.

(*b*) To find the m. s. e. of an angle in the adjusted figure.

$$\text{Angle} = NLS.$$
$$\therefore dF = -(2) - (5)$$
$$= +0.055 \text{ I.} - 0.067 \text{ II.} + 0.700 \text{ III.}$$

from the weight equations.

From equations (25), Art. 130,

$$q_1 = [\alpha\alpha]g_1 + [\alpha\beta]g_2 + \cdots$$
$$q_2 = [\alpha\beta]g_1 + [\beta\beta]g_2 + \cdots$$
$$\cdots\cdots\cdots\cdots$$

The values of $[\alpha\alpha]$, $[\alpha\beta]$ . . . are given in the weight equations. Hence,

$$q_1 = +0.267 \times 0 - 0.233 \times -1 = +0.233,$$
$$q_2 = -0.233 \times 0 + 0.267 \times -1 = -0.267,$$
$$q_3 = +0.038 \times 0 - 0.021 \times -1 = +0.021,$$
$$q_4 = -0.021 \times 0 + 0.088 \times -1 = -0.088.$$

g	q	gq
0	+ 0.233	0
− 1	− 0.267	0.267
0	+ 0.021	0
− 1	− 0.088	0.088

$$\frac{[gA]^2}{[aA]} = 0.022.$$

$$\frac{[gB.1]^2}{[bB.1]} = 0.006$$

$$\frac{[gC.2]^2}{[cC.2]} = \underline{0.274}$$

$$0.302$$

(See the solution of the normal equations.)

$$[gq] = 0.355$$
$$-\underline{0.302}$$
$$u_F = +0.053$$
$$\therefore \mu_F = 1.23'' \sqrt{0.053}$$
$$= 0.28''.$$

(*c*) To find the m. s. e. of a side in the adjusted figure.

Side = Oneota–Lester.

$$F = OL = NS \frac{\sin ONS}{\sin SON} \frac{\sin OSL}{\sin OLS}.$$

Therefore,

$$dF = 1.33\,(1) + 1.33\,(2) + 0.74\,(4) + 0.74\,(5) - 3.04\,(7) - 3.52\,(9)$$

in units of the sixth place of decimals,

$$= -0.174\ \text{I.} - 0.475\ \text{II.} + 0.015\ \text{III.}$$

from the weight equations.

The solution is carried through exactly as in the preceding case. We find,

$$[gq] = 2.011 \quad \text{and} \quad u_F = 1.49.$$

Hence, $\mu\sqrt{u_F} = 1.23\sqrt{1.49}$

$= 1.5$ in units of the sixth place of decimals.

Now, $\log OL = 4.2189699 \qquad OL = 16556\ m.$

$$\therefore \text{ m. s. e. of side} = \frac{16556}{0.434} \times 0.0000015$$

$$= 0.06\ m.$$

Solution of the Normal Equations.

I.	II.	III.	l	f (Angle).
+ 0.138	+ 0.049 + 1.241 . . .	+ 0.021 + 1.687 + 4.706	− 0.260 − 1.540 − 2.970	+ 0.055 − 0.067 + 0.700
+ 1	+ 0.355 + 1.224	+ 0.152 + 1.680 + 4.703 . . .	− 1.885 − 1.448 − 2.930 . . .	+ 0.399 − 0.087 + 0.692 + 0.022
. . . .	+ 1	+ 1.373 + 2.396 . . .	− 1.183 − 0.945 . . .	− 0.071 + 0.811 + 0.006
. . . . . .	. . . . . .	+ 1 . . .	− 0.394 . . .	+ 0.338 + 0.274

162. Ex. 1. — Adjust the observed differences of longitude * given in the following table:

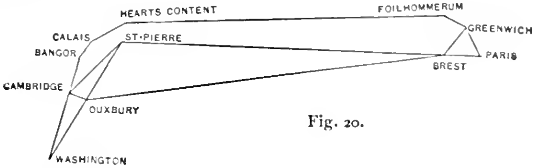

Fig. 20.

Dates.		Observed Differences.				Corrections.
		h.	m.	s.	s.	
1851	Cambridge–Bangor	0	9	23.080	± 0.043	v_1
1857	Bangor–Calais		6	00.316	± 0.015	v_2
1866	Calais–Heart's Content . . .		55	37.973	± 0.066	v_3
1866	Heart's Content–Foilhommerum	2	51	56.356	± 0.029	v_4
1866	Foilhommerum–Greenwich . .		41	33.336	± 0.049	v_5
1872	Brest–Greenwich		17	57.598	± 0.022	v_6
1872	Brest–Paris		27	18.512	± 0.027	v_7
1872	Greenwich–Paris		9	21.000	± 0.038	v_8
1872	St. Pierre–Brest	3	26	44.810	± 0.027	v_9
1872	Cambridge–St. Pierre		59	48.608	± 0.021	v_{10}
1869–1870	Cambridge–Duxbury		1	50.191	± 0.022	v_{11}
1870	Duxbury–Brest	4	24	43.276	± 0.047	v_{12}
1867 } 1872 }	Washington–Cambridge . . .		23	41.041	± 0.018	v_{13}
1872	Washington–St. Pierre . . .		23	29.553	± 0.027	v_{14}

[Number of conditions $= n - s + 1$, where n is number of observed differences of longitude, and s is number of longitude stations.

The condition equations are

$$
\begin{aligned}
&& s. \\
-v_6 + v_7 - v_8 &= &+0.086 \\
-v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_9 + v_{10} &= &+0.045 \\
-v_9 - v_{10} + v_{11} + v_{12} &= &-0.049 \\
+v_{10} + v_{13} - v_{14} &= &-0.096
\end{aligned}
$$

The weights are taken inversely as the squares of the p e.

Solution by method of correlates, as in Art. 119.]

Ex. 2. — The system of triangulation shown in the figure was executed by Koppe in the determination of the axis (Airolo-Göschenen) of the St.

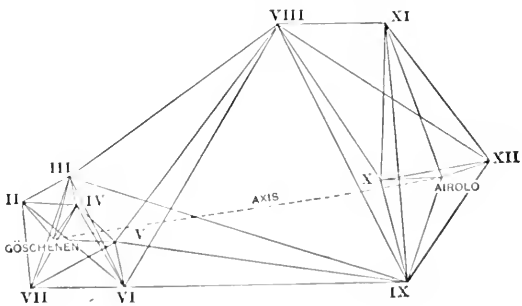

Fig. 21.

* *Coast and Geodetic Survey Report*, 1880, app. No. 6.

Gothard tunnel.* In the following table the adjusted values are given side by side with the measured values. It is proposed as a problem of adjustment.

At Göschenen.

	Measured. °	′	″	Adjusted. ″
II.	0	00	00.00	00.00
III.	44	33	10.88	10.03
IV.	69	30	12.51	11.62
V.	124	58	4.23	5.13

At II.

	°	′	″	″
III.	0	00	00.00	00.00
IV.	37	53	54.33	52.97
V.	60	29	33.13	33.82
VI.	77	4	5.67	8.17
Göschenen	93	11	41.69	40.57
VII.	124	16	33.98	33.27

At III:

	°	′	″	″
VIII.	0	00	00.00	00.00
IX.	53	58	14.48	15.49
VI.	99	47	50.21	50.86
IV.	102	32	51.36	51.90
Göschenen	138	44	28.81	29.70
VII.	144	28	12.47	11.40
II.	180	59	38.94	39.11

At VIII.

	°	′	″	″
XI.	0	00	00.00	00.00
XII.	18	56	17.43	17.54
X.	43	50	24.03	24.70
IX.	50	18	22.52	20.27
VI.	106	30	15.04	15.37
V.	112		28.72	29.24
III.	130	11	30.81	41.54

At IX.

	°	′	″	″
VI.	0	00	00.00	00.00
V.	8	28	17.13	15.06
III.	18	33	3.27	5.00
VIII.	63	41	28.63	28.55
X.	76	59	50.89	51.48
XI.	79	10	36.33	36.34
Airolo	109	45	39.23	39.33
XII.	123	16	23.76	24.23

At IV.

	Measured. °	′	″	Adjusted. ″
V.	0	00	00.00	00.00
VI.	15	41	3.57	6.29
VII.	74	12	20.55	19.86
Göschenen	80	32	48.99	50.12
II.	135	44	49.77	50.91
III.	199	24	11.56	10.73

At V.

	°	′	″	″
IV.	0	00	00.00	00.00
VIII.	78	40	5.91	6.72
IX.	140	44	43.51	44.45
VI.	215	32	45.41	43.45
VII.	286	19	25.30	27.21
Göschenen	316	00	44.92	43.61
II.	338	20	33.53	31.74

At VII.

	°	′	″	″
II.	0	00	00.00	00.00
III.	19	11	58.44	59.03
IV.	32	4	49.32	48.68
V.	64	11	54.08	56.05
VI.	90	05	39.47	37.00

At XI.

	°	′	″	″
XII.	0	00	00.00	00.00
Airolo	16	55	55.06	54.38
IX.	37	13	59.79	58.43
VIII.	152	26	30.24	30.44

At XII.

	°	′	″	″
IX.	0	00	00.00	00.00
Airolo	30	31	2.30	3.39
X.	42	13	20.53	21.33
VIII.	90	3	2.22	1.74
XI.	98	40	14.95	13.72

At Airolo.

	°	′	″	″
XI.	0	00	00.00	00.00

* *Zeitschr. für Vermess.*, vol. iv.

At X.				
XII.	0	00	00.00	00.00
Airolo	9	49	30.02	37.92
IX.	91	30	5.16	5.96
VIII.	252	43	46.75	47.49
XI.	275	12	8.44	9.74
XII.	94	54	56.06	55.26
IX.	230	53	7.51	6.98
X.	290	26	49.43	51.11

The distance X–XII is 4416.8 *m.*
There are 19 angle equations and 15 side equations in the adjustment.

Solution by Groups.

163. The rigorous forms of solution which have been given are suitable for a primary triangulation where the greatest accuracy is required. In secondary or tertiary work it is frequently not advisable to spend so much labor in the reduction. For work of this kind the group method of solution is to be preferred.

The solution by groups may be made by either of two general methods. First, each condition or set of conditions may be adjusted for independently in succession, the values of the corrections found at each adjustment being closer and closer approximations to the final values. Should the values found, after going through all of the conditions, not satisfy the first and second groups of condition equations closely enough, the process must be repeated until the required accuracy is attained. This is the method outlined in general terms in Art. 131.

Second, each group may be adjusted in turn while the results of the adjustment of the preceding groups are preserved by insuring that the conditions which have been satisfied remain so. This is an exact method.

164. To make the operation as simple as possible, let us take but a single condition at a time.

(1) Local equation at N. Base,

$$v_1 + v_2 + v_3 + 1.37 = 0.$$

The solution is given in Ex. 1, Art. 139,

$$v_1 = -0.64'', \ v_2 = -0.64'', \ v_3 = -0.09.$$

(2) Local equation at S. Base,

$$v_4 + v_5 - v_6 + 1.07 = 0.$$

The solution is given in Ex. 2, Art. 139,

$$v_4 = -0.13'',\ v_5 = -0.50'',\ v_6 = +0.44''.$$

(3) Angle equation,

$$v_3 + v_4 + v_7 + 0.48 = 0.$$

Using the values of v_3, v_4, already found as first approximations, the equation reduces to

$$v_3 + v_4 + v_7 + 0.26 = 0.$$

The method of solution is given in Ex. 2, Art. 120,

$$v_3 = -0.13'',\ v_4 = -0.08'',\ v\ = -0.05''.$$

The successive approximations found so far, when added, give

$$\begin{aligned} v_1 &= -0.64'', & v_5 &= -0.55'', \\ v_2 &= -0.64'', & v_6 &= +0.44'', \\ v_3 &= -0.22'', & v_7 &= -0.05''. \\ v_4 &= -0.21'', \end{aligned}$$

Proceed similarly with the remaining two condition equations. The resulting values will agree closely with the rigorous values already found.

165. In order to bring out still more clearly the advantages of solving in this way, let us take a more extended example. A good one is furnished by the triangulation (1874–1878) of the east end of Lake Ontario, omitting the system around the Sandy Creek base.

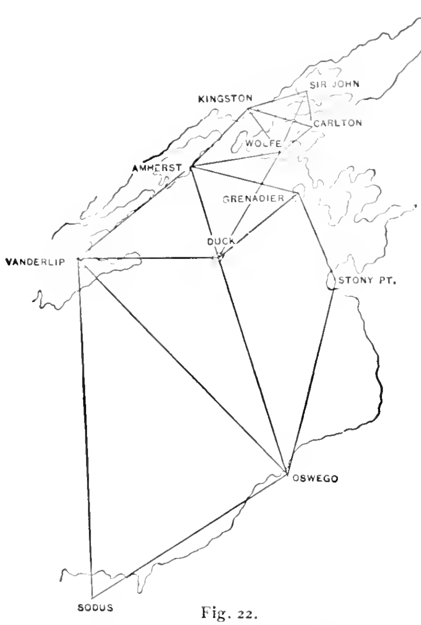

Fig. 22.

The measured values of the angles are given in the following table. Each angle is taken to be of the same weight. In the last column are given the locally corrected angles found by the rigorous methods of solution.

Station Occupied.	Angle as Measured Between	°	′	″	Locally Corr. Angles. ″
Sir John . .	Carlton and Kingston . .	90	17	44.01	. . .
	Wolfe and Kingston. . .	56	24	09.77	. . .
Carlton . .	Wolfe and Sir John . . .	120	48	06.54	. . .
	Kingston and Sir John. .	62	03	27.56	. . .
Kingston. .	Sir John and Wolfe . . .	64	40	50.91	. . .
	Carlton and Wolfe . . .	37	02	04.43	. . .
	Wolfe and Amherst . . .	88	19	14.70	. . .
Wolfe . . .	Duck and Carlton	188	07	18.54	. . .
	Amherst and Carlton . .	140	12	34.44	. . .
	Kingston and Carlton . .	84	13	14.34	. . .
	Sir John and Carlton . .	25	18	16.80	. . .

Station Occupied.	Angle as Measured Between	°	′	″	Locally Corr. Angles. ″
Amherst . .	Kingston and Wolfe. . .	35	41	23.02	22.69
	Kingston and Duck . . .	111	45	28.46	28.68
	Wolfe and Duck	76	04	06.32	05.99
	Grenadier and Duck. . .	54	38	00.34	. . .
	Duck and Vanderlip. . .	71	15	25.43	25.32
	Vanderlip and Kingston .	176	59	06.11	06.00
Duck . . .	Oswego and Vanderlip. .	104	08	58.93	59.10
	Vanderlip and Amherst .	70	26	31.99	32.16
	Amherst and Wolfe . . .	56	01	12.47	12.64
	Wolfe and Grenadier . .	18	45	43.36	43.53
	Grenadier and Stony Pt. .	49	53	12.77	12.94
	Stony Pt. and Oswego . .	60	44	19.46	19.63
Grenadier .	Stony Pt. and Duck . . .	78	13	33.64	33.84
	Duck and Amherst . . .	50	35	04.28	04.19
	Duck and Stony Pt. . . .	281	46	25.89	26.16
	Amherst and Stony Pt. .	231	11	22.04	21.97
Stony Point.	Oswego and Duck. . . .	88	22	00.86	. .
	Duck and Grenadier. . .	51	53	12.60	12 70
	Grenadier and Duck. . .	308	06	47.21	47.30
Oswego . .	Sodus and Vanderlip . .	80	29	46 10	46.59
	Sodus and Duck	107	19	03.28	03.96
	Sodus and Stony Pt.. . .	138	12	49.44	48.28
	Vanderlip and Duck. . .	26	49	16.61	17.37
	Vanderlip and Stony Pt..	57	43	01.96	01.69
	Duck and Stony Pt. . . .	30	53	42.88	44.32
Vanderlip .	Amherst and Duck . . .	38	18	07.12	07.30
	Amherst and Oswego . .	87	19	53.47	53.16
	Duck and Oswego. . . .	49	01	45.54	45.86
	Duck and Sodus	87	59	12.55	12.42
	Oswego and Sodus . . .	38	57	26.55	26.56
	Sodus and Amherst . . .	233	42	40.41	40.28
Sodus . . .	Vanderlip and Oswego. .	60	32	57.55	. . .

The local and general equations are formed as usual (see Arts. (133–155). The general rule in the solution is to adjust for one condition at a time. Instead, however, of following out this rule strictly, it is often better to adjust for a *group* of conditions simultaneously. Often a group is almost as easily managed as a single condition. No rule can be given to cover all cases, and much must be left to the judgment and ingenuity of the computer.

166. The Local Adjustment at Each Station.

(*a*) Adjust for each sum angle separately.

Rule and example in Arts. 138–139.

(*b*) Adjust for closure of the horizon.

Rule and example in Arts. 138–139.

At stations Sir John, Carlton, Kingston, Wolfe, there are no local conditions; and at each of the stations Amherst, Stony Point, Sodus, there is one angle independent of the others, and therefore not locally adjusted.

The angles at station Amherst may be rigorously adjusted, as in Art. 138. The resulting values are given in the table. If we break the adjustment into two parts, as in (*a*) and (*b*), we have:

(*a*) *Sum Angle.*

	Measured Values.				Adjusted.
	°	′	″	″	″
Kingston–Wolfe,	35	41	23.02	− 0.29	22.73
Wolfe–Duck,	76	04	06.32	− 0.29	06.03
	111	45	29.34		28.76
Kingston–Duck,	111	45	28.46	+ 0.29	28.75 check.
			3) 0.88		
			0.29		

(*b*) *Closure of Horizon.*

	°	′	″	″	″
Kingston–Wolfe,	35	41	22.73	− 0.08	22.65
Wolfe–Duck,	76	04	06.03	− 0.07	05.96
Duck–Vanderlip,	71	15	25.43	− 0.08	25.35
Vanderlip–Kingston,	176	59	06.11	− 0.07	06.04
			4) 00.30		00.00 check.
			00.075		

The adjusted values agree closely with those from the simultaneous solution, as given in the table.

At station Duck the angles close the horizon. Hence the correction to each angle is one-sixth of the difference of their sum from 360°. (See Art. 139.)

167. The General Adjustment. — The local adjustment being finished, we shall consider the adjusted angles to be inde-

pendent of one another and to be of the same weight. We are therefore at liberty to break up the net into its simplest parts. We have in our figure, first a quadrilateral *SCWK*, next two single triangles *KWA*, *AWD*, next a central polygon *DAGS* and, lastly, a single triangle *VOS*. These three figures include most cases that arise in any triangulation net.

168. (*a* 1) **Adjustment of a Quadrilateral: Approximate Method.** — In the quadrilateral *SCKW* all of the eight angles 1, 2, . . . 8 are supposed to be equally well measured.

(1) *The Angle Equations.*

The angle equations from the triangles *SCW, CWK, WKS,* may be written in general terms

$$v_1 + v_2 + v_3 + v_4 = l_1,$$
$$v_3 + v_4 + v_5 + v_6 = l_2,$$
$$v_5 + v_6 + v_7 + v_8 = l_3.$$

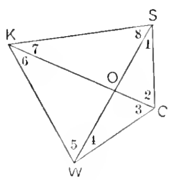

Fig. 23.

As these equations are entangled, if we adjusted for each in succession a great many repetitions of the adjustment would be necessary to obtain values that would satisfy the equations simultaneously. It is, therefore, better to adjust simultaneously, and it happens that a very simple rule for doing this can be found.

Call C_1, C_2, C_3, C_4, the correlates of the equations in order; then the correlate equations are

$$\begin{aligned} C_1 \qquad &= v_1, \qquad & C_2 + C_3 &= v_5, \\ C_1 \qquad &= v_2, & C_2 + C_3 &= v_6, \\ C_1 + C_2 &= v_3, & C_3 &= v_7, \\ C_1 + C_2 &= v_4, & C_3 &= v_8, \end{aligned}$$

and the normal equations,

$$\begin{aligned} 4\,C_1 + 2\,C_2 \qquad\quad &= l_1, \\ 2\,C_1 + 4\,C_2 + 2\,C_3 &= l_2, \\ 2\,C_2 + 4\,C_3 &= l_3. \end{aligned}$$

Solving these equations, there result,

$$\begin{aligned}
C_1 &= \tfrac{1}{8}\,(+\,3\,l_1 - 2\,l_2 + \ \ l_3),\\
C_2 &= \tfrac{1}{8}\,(-\,2\,l_1 + 4\,l_2 - 2\,l_3),\\
C_3 &= \tfrac{1}{8}\,(+\ \ l_1 - 2\,l_2 + 3\,l_3).
\end{aligned}$$

Substitute these values in the correlate equations, and

$$\begin{aligned}
v_1 = v_2 &= \tfrac{1}{8}\,(+\,3\,l_1 - 2\,l_2 + \ \ l_3),\\
v_3 = v_4 &= \tfrac{1}{8}\,(-\ \ l_1 + 2\,l_2 - \ \ l_3),\\
v_5 = v_6 &= \tfrac{1}{8}\,(-\ \ l_1 + 2\,l_2 + \ \ l_3),\\
v_7 = v_8 &= \tfrac{1}{8}\,(+\ \ l_1 - 2\,l_2 + 3\,l_3),
\end{aligned}$$

which may be written,

$$\begin{aligned}
v_1 = v_2 &= \tfrac{1}{4}\,l_1 - \tfrac{1}{4}\,(l_2 - \tfrac{1}{2}\,l_3 - \tfrac{1}{2}\,l_1),\\
v_3 = v_4 &= \tfrac{1}{4}\,l_1 + \tfrac{1}{4}\,(l_2 - \tfrac{1}{2}\,l_3 - \tfrac{1}{2}\,l_1),\\
v_5 = v_6 &= \tfrac{1}{4}\,l_3 + \tfrac{1}{4}\,(l_2 - \tfrac{1}{2}\,l_3 - \tfrac{1}{2}\,l_1),\\
v_7 = v_8 &= \tfrac{1}{4}\,l_3 - \tfrac{1}{4}\,(l_2 - \tfrac{1}{2}\,l_3 - \tfrac{1}{2}\,l_1),
\end{aligned}$$

whence follows at once the convenient rule for adjusting the quadrilateral, so far as the angle equations are concerned:

(α) *Write the measured angles in order of azimuth in two sets of four each, the first set being the angles of SCW, and the second those of WKS.*

(β) *Adjust the angles of each set by one-fourth of the difference of this sum from* 180° + *excess of triangle, arranging the adjusted angles in two columns, so that the first column will show the angles of SCK, and the second those of CWK.*

(γ) *Adjust the first column by one-fourth of the difference of its sum from* 180° + *excess of triangle, and apply the same correction, with the sign changed, to the second column.*

This rule was first published by me in the *Journal of the Franklin Institute*, June, 1880.

The spherical excesses of the triangles *SCW*, *CWK*, *WKS*, being 0.16″, 0.35″, and 0.47″, respectively, the adjustment of the quadrilateral may be arranged as follows:

Measured Angles.						Adjusted Angles.
	°	′	″	″		″
	33	53	35.14	35.56		35.40
	62	03	27.56	27.98		27.82
	58	44	38.98		39.40	39.56
	25	18	16.80		17.22	17.38
	179	59	58.48			00.16 check
$180 + \epsilon =$	180	00	00.16			
			4) 1.68			
			0.42			
	58	54	57.54		58.11	58.27
	37	02	04.43		04.99	05.15
	27	38	46.48	47.04		46.88
	56	24	9.77	10.33		10.17
	179	59	58.22	00.91		00.47 check
$180 + \epsilon =$	180	00	00.47	00.28		
			4) 2.25	4) 0.63		
			0.56	0.16		

169. (2) *The Side Equation.*

Using the values of the angles just found, we next form the side equation with pole at O. It is

$$\frac{\sin OSC}{\sin SCO}\,\frac{\sin OCW}{\sin CWO}\,\frac{\sin OWK}{\sin WKO}\,\frac{\sin OKS}{\sin KSO} = 1\,;$$

or writing it in general terms, when reduced to the linear form,

$$a_1v_1' + a_2v_2' + a_3v_3' + a_4v_4' + a_5v_5' + a_6v_6' + a_7v_7' + a_8v_8' = l_4,$$

where v_1', v_2', . . . are the corrections resulting from the side equation.

Solving as in Ex. 2, Art. 120, we have the corrections

$$v_1' = \frac{a_1}{[aa]}l_4,\ v_2' = \frac{a_2}{[aa]}l_4,\ \ldots$$

These corrections may be found still more rapidly as follows: Since the side equation may be so transformed that the coefficients a_1, a_2, . . . are approximately equal to unity numerically

(see Art. 149), we may take each of them to be unity, and then

$$v_1' = v_3' = v_5' = v_7' = +\tfrac{1}{8}\, l_4,$$
$$v_2' = v_4' = v_6' = v_8' = -\tfrac{1}{8}\, l_4;$$

that is, *the corrections to the angles are numerically equal, but are alternately + and −.*

This plan has the additional advantage of not disturbing the angle equations. The rule gives approximate results which are the more nearly correct, the more nearly the coefficients a_1, a_2, . . . are equal to each other and the smaller is the absolute term of the side equation. In many cases it will give results which depart widely from those found by the exact process.

Returning to our numerical example, we first reduce the side equation to the linear form

$$\begin{array}{llll}
 & \;\;\circ \;\;\; ' \;\;\;\; '' & & \;\;\circ \;\;\; ' \;\;\;\; '' \\
OSC &= 33\ 53\ 35.40 + v_1, & SCO &= 62\ 03\ 27.82 + v_2, \\
OCW &= 58\ 44\ 39.56 + v_3, & CWO &= 25\ 18\ 17.38 + v_4, \\
OWK &= 58\ 54\ 58.27 + v_5, & WKO &= 37\ 02\ 05.15 + v_6, \\
OKS &= 27\ 38\ 46.88 + v_7, & KSO &= 56\ 24\ 10.17 + v_8.
\end{array}$$

$$\begin{array}{ll}
9.7463587 + 31.3\, v_1 & 9.9461673 + 11.2\, v_2 \\
9.9318952 + 12.8\, v_3 & 9.6308691 + 44.5\, v_4 \\
9.9326832 + 12.7\, v_5 & 9.7797125 + 27.9\, v_6 \\
9.6665301 + 40.2\, v_7 & 9.9206181 + 14.0\, v_8 \\
\qquad\quad 72 & \qquad\quad 70 \\
\qquad\quad 70 & \\
\qquad\quad\;\, 2 &
\end{array}$$

Dividing by 20, which will reduce the coefficients to unity approximately, and

$$1.56\, v_1' - 0.56\, v_2' + 0.64\, v_3' - 2.22\, v_4' + 0.64\, v_5'$$
$$- 1.40\, v_6' + 2.01\, v_7' - 0.70\, v_8' + 0.10 = 0.$$

Hence, $\qquad [aa] = 15,$

and $\qquad v_1' = -\,0.01'', \quad v_2' = 0.00'', \quad v_3' = 0.00'', \quad v_4' = +\,0.01''$, etc.

By the second rule the corrections would be $\mp \dfrac{0.1''}{8}$, that is, $\mp\, 0.01''$, alternately, which values differ but little from the preceding.

The total corrections to the angles are the sums of the two sets of corrections from the angle and side equations.

170. (*a* 2) **Adjustment of a Quadrilateral: Rigorous Method.**—By the following artifice the quadrilateral may be *rigorously*

adjusted for the side equation without disturbing the angle equation adjustment, which amounts to the same thing as the simultaneous adjustment of the angle and side equations.

Suppose that the angle equations have been adjusted as already explained in (a 1). If v_1', v_2', . . . v_8', denote the corrections arising from the side equation, the condition equations may be written

$$
\begin{aligned}
v_1' + v_2' + v_3' + v_4' &= 0, \\
v_3' + v_4' + v_5' + v_6' &= 0, \\
v_5' + v_6' + v_7' + v_8' &= 0, \\
a_1 v_1' + a_2 v_2' + a_3 v_3' + a_4 v_4' + a_5 v_5' + a_6 v_6' + a_7 v_7' + a_8 v_8' &= l_4'.
\end{aligned}
$$

By writing the corrections in the form

$$
\begin{aligned}
v_1' &= + v + v', & v_5' &= + v + v''', \\
v_2' &= + v - v', & v_6' &= + v - v''', \\
v_3' &= - v + v'', & v_7' &= - v + v'''', \\
v_4' &= - v - v'', & v_8' &= - v - v'''',
\end{aligned}
$$

the first three condition equations become $0 = 0$ identically, and we have therefore to deal only with the single condition equation

$$
\begin{aligned}
(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)\, v + (a_1 - a_2)\, v' \\
+ (a_3 - a_4)\, v'' + (a_5 - a_6)\, v''' + (a_7 - a_8)\, v'''' = l_4',
\end{aligned}
$$

with

$$(v + v')^2 + (v - v')^2 + (- v + v'')^2 + (- v - v'')^2 + \cdots = \text{a min.}$$

The correlate equations are,

$$
\begin{aligned}
(a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)\, C &= 4\, v, \\
(a_1 - a_2)\, C &= v', \\
(a_3 - a_4)\, C &= v'', \\
(a_5 - a_6)\, C &= v''', \\
(a_7 - a_8)\, C &= v''''.
\end{aligned}
$$

Substitute in the condition equation, and

$$
\begin{aligned}
C \{ \tfrac{1}{4} (a_1 + a_2 + a_5 + a_6 - a_3 - a_4 - a_7 - a_8)^2 + (a_1 - a_2)^2 \\
+ (a_3 - a_4)^2 + (a_5 - a_6)^2 + (a_7 - a_8)^2 \} = l_4'
\end{aligned}
$$

from which C can be found.

Hence the corrections are known.

The complete adjustment of our quadrilateral is contained in the following table :

Meas. Angles.					Local Angles.	Log Sines.		Log Diff.		Sums.	Sqs.
°		″	″	″	″						
33	53	35.14	35.56		35.40	9.7463587		+ 31.3			
62	03	27.56	27.98		27.82		9.9461673		+ 11.2	42.5	1806
58	44	38.98		39.40	39.56	9.9318952			+ 12.8		
25	18	16.80		17.22	17.38		9.6308691	+ 44.5		57.3	3283
		58.48									
		00.16									
		1.68									
58	54	57.54		58.11	58.27	9.9326832		+ 12.7			
37	02	04.43		04.99	05.15		9.7798125		+ 27.9	40.6	1648
27	38	46.48	47.04		46.88	9.6665301			+ 40.2		
56	24	09.77	10.33		10.17		9.9206181	+ 14.0		54.2	2938
		58.22	00.91								
		00.47	00.28			72	70	102.5	92.1		
						70		92.1			
		2.25	0.63					4)10.04		5.2	
						2		2.6			27
											9702

$$\therefore \frac{v'}{2.6} = \frac{v'}{42.5} = \frac{v''}{57.3} = \frac{v'''}{40.6} = \frac{v''''}{54.2} = -\frac{2}{9702}.$$

Hence the corrections are known. These corrections, applied to the local angles, give the final angles required.

171. (*b*) Adjustment of a Single Triangle.

Rule and example in Ex. 2, Art. 118, and Ex. 2, Art. 120.

The single triangles in our figure are Kingston, Wolfe, Amherst ; Wolfe, Duck, Amherst ; Vanderlip, Oswego, Sodus.

For example, take the first ($\epsilon = 0.72''$):

	Measured.			Adjusted.	
	°	′	″	″	
Kingston . .	88	19	14.70	15.78	
Wolfe . . .	55	59	20.10	21.18	
Amherst . .	35	41	22.69	23.77	
	179	59	57.49	0.73	check.
$180 + \epsilon =$	180	00	00.72		
			3) 3.23		
			1.08		

172. (*c*) **Adjustment of a Central Polygon.** — In the central polygon Duck, Amherst, Grenadier, Stony Point, Oswego, Vanderlip, the condition equations in general terms are:

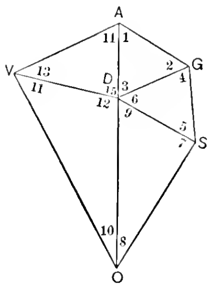

Fig. 24.

Local equation (horizon equation),

$$v_3 + v_6 + v_9 + v_{12} + v_{15} = l_1.$$

Angle equations,

$$v_1 + v_2 + v_3 = l_2,$$
$$v_4 + v_5 + v_6 = l_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$v_{13} + v_{14} + v_{15} = l_6.$$

Side equation (pole at Duck),

$$a_1 v_1 + a_2 v_2 + a_4 v_4 + a_5 v_5 + \cdot \cdot \cdot + a_{14} v_{14} = l.$$

We may adjust for these equations in order, first the horizon equation, then the angle equations separately, as they are not entangled, and next the side equation.

A rigorous adjustment may, however, be carried out at once with very little additional labor. Adjust first each angle equation by itself, and let (v_1), (v_2), . . . be the values that result. Let (1), (2), . . . denote the farther corrections to the measured angles in order arising from the local and side equations, so that

$$v_1 = (v_1) + (1),$$
$$v_2 = (v_2) + (2),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If we substitute these values in the above equations we have the new condition equations,

$$a_1 (1) + a_2 (2) + \cdot \cdot \cdot + a_{14} (14) = l',$$
$$(3) + (6) + (9) + (12) + (15) = l'',$$
$$(1) + (2) + (3) = 0'',$$

$$(4) + (5) + (6) = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$(13) + (14) + (15) = 0,$$

from which to find (1), (2), (3), . . . (15).

Calling C_1, C_2, I, II, . . . the correlates of the condition equations in order, we have the correlate equations

$$\begin{array}{lll} (1) = a_1C_1 + I, & (4) = a_4C_1 + II & \cdots\cdots \\ (2) = a_2C_1 + I, & (5) = a_5C_1 + II & \cdots\cdots \\ (3) = \quad C_2 + I, & (6) = \quad C_2 + II & \cdots\cdots \end{array}$$

Eliminate now the angle equation correlates. By addition,

$$\begin{aligned} (a_1 + a_2)\,C_1 + 3\,I \;\, + C_2 &= 0, \\ (a_4 + a_5)\,C_1 + 3\,II + C_2 &= 0, \\ \cdots\cdots\cdots\cdots \end{aligned}$$

Hence

$$\begin{aligned} (1) &= + \tfrac{1}{3}\,(2\,a_1 - \quad a_2)\,C_1 - \tfrac{1}{3}\,C_2, \\ (2) &= - \tfrac{1}{3}\,(\quad a_1 - 2\,a_2)\,C_1 - \tfrac{1}{3}\,C_2, \\ (3) &= - \tfrac{1}{3}\,(\quad a_1 + \quad a_2)\,C_1 + \tfrac{2}{3}\,C_2, \\ \cdots\cdots\cdots\cdots \end{aligned}$$

Substitute these values of (1), (2), . . . in the condition equations, and we have the normal equations

$$\begin{aligned} 2\,\{[aa] - a_1a_2 - a_4a_5 - \cdots\}\,C_1 - [a]\,C_2 &= 3\,l', \\ -[a]\,C_1 + 10\,C_2 &= 3\,l'', \end{aligned}$$

Solving, we find C_1, C_2, and thence (1), (2), . . . are known. The normal equations may be written down directly in every case, as the law of their formation is as evident as that of ordinary normal equations. They involve only two unknowns, C_1, C_2, no matter how many sides the polygon has.

173. We shall now proceed with our numerical example.

At station Duck the measured values of the angles are taken, at the other stations the locally adjusted angles.

Given Angles.		Log Sines.		Dif. 1″.	Squares.	Products.	Sums.
° ′ ″	″						
54 38 00.34	00.62	9.9114060		+14.9	222.0	257.8	− 2.4
50 35 04.19	04.47		9.8879338	+17.3	299.3		
74 46 55.83	56.12						
180 00 00.36							
180 00 01.21 = 180 + ε							
3) 0.85							
0.28							
78 13 33.84	34.49	9.9907654		+ 4.4	19.4	72.6	−12.1
51 53 12.70	13.35		9.8958618	+16.5	272.2		
49 53 12.77	13.42						
59.31							
1.26							
1.95							
0.65							
88 22 00.86	00.47	9.9998235		+ 0.6	.4	21.1	−34.6
30 53 44.32	43.93		9.7105188	+35.2	1239.0		
60 44 19.46	19.07						
04.64							
3.47							
1.17							
0.39							
26 49 17.37	18.15	9.6543842		+41.6	1730.6	761.3	+23.3
49 01 45.86	46.64		9.8779750	+18.3	334.9		
104 08 58.93	59.70						
2.16							
4.49							
2.33							
0.78							
38 18 07.30	06.33	9.7922537		+26.7	712.9	189.6	+19.6
71 15 25.32	24.36		9.9763353	+ 7.1	50.4		
70 26 31.99	31.02						
04.61		6328	6247		4881.1	1302.4	− 6.2
1.71		6247			1302.4		
2.90		81			6183.5		
0.97		3			2		
		243			12,367.0		

The Normal Equations.

$$12{,}367\, C_1 + 6.2\, C_2 = -243$$
$$6.2\, C_1 + 10\, C_2 = 2.01$$

$$\therefore\ C_1 = -0.020$$
$$C_2 = +0.213$$

Local Equation at Station Duck.

°	′	″
74	46	56.12
49	53	13.42
60	44	19.07
104	08	59.70
70	26	31.02
359	59	59.33
360	00	00.00
		00.67
		3
		2.01

Corrections.	Adjusted Angles.		
$''$	°	′	″
(1) = − 0.39	54	38	00.23
(2) = + 0.26	50	35	04.73
(3) = + 0.13	74	46	56.25
(4) = − 0.24	78	13	34.25
(5) = + 0.18	51	53	13.53
(6) = + 0.06	49	53	13.48
(7) = − 0.31	88	22	00.16
(8) = + 0.40	30	53	44.33
(9) = − 0.09	60	44	18.98
(10) = − 0.75	26	49	17.40
(11) = + 0.45	49	01	47.09
(12) = + 0.30	104	08	60.00
(13) = − 0.47	38	18	05.86
(14) = + 0.20	71	15	24.56
(15) = + 0.27	70	26	31.29

Approximate Method of Finding the Precision.

174. An adjustment may be carried out rigorously so far as finding the values of the unknowns is concerned, but only an approximate value of the m. s. e. of the angles or sides may be thought necessary.

In good work the following method will give results nearly the same as those found by the rigorous process.

The average value μ' of the m. s. e. of an angle in a triangulation net after adjustment is easily seen from Art. 113 to be

$$\mu' = \sqrt{\frac{n - n_c}{n}}\,\mu_m,$$

where

n = number of angles observed,
n_c = number of local and general conditions,
μ_m = m. s. e. of a measured angle of average weight;

or, if all the angles are of equal weight, it is the m. s. e. of a measured angle.

The value of μ, the m. s. e. of an angle of unit weight, is, by the usual formula,

$$\mu = \sqrt{\frac{[pv^2]}{n_c}},$$

$$\mu_m = \frac{\mu}{\sqrt{p_m}},$$

in which p_m is the average weight of a measured angle.

To find the m. s. e. of a side of a triangle, a single chain of the best-shaped triangles between the base and the side is selected, all tie lines being rejected. Then, assuming the base to be exact and the m. s. e. of each adjusted angle to be μ', we have from Ex. 9, Art. 126,

$$\mu^2_{\log an} = \tfrac{2}{3}\,\mu'^2\,[\delta_A{}^2 + \delta_B{}^2 + \delta_A\delta_B],$$

where δ_A, δ_B are the log differences corresponding to $1''$ for the angles A, B in a table of log sines.

Ex.—To find the m. s. e. of the side OL as derived from the base NS in the figure $ONSL$ (Fig. 19).

Number of angles measured = 9.
Number of conditions, local and general, = 5.
From the adjustment (Art. 157) $[pv^2] = 7.54$.

$$\therefore \mu = \sqrt{\frac{7.54}{5}}$$

$$= 1.23'',$$

$$p_m = 10.4 \text{ and } \therefore \mu_m = \frac{1.23}{\sqrt{10.4}} = 0.38$$

$$\mu' = \sqrt{\frac{9-5}{9}}\,\mu_m$$

$$= \tfrac{2}{3}\,\mu_m$$

$$= 0.26''.$$

CHAPTER VII

APPLICATION TO THE ADJUSTMENT OF A TRIANGULATION. METHOD OF DIRECTIONS

175. This method is due to Bessel. Various modifications of Bessel's plan of making the observations are in use on different surveys. The following is that used in the Coast and Geodetic Survey on primary triangulation at the present time.

Each series of observations consists of successive pointings on the various stations in order, from left to right, with corresponding readings of the horizontal circle with three micrometer microscopes, followed immediately by pointings on the same stations in the reverse order, after reversing the position of the horizontal axis of the telescope in the wyes, and turning the alidade 180° in azimuth, each pivot remaining in contact with the same wye as before. Each observation of an angle consists therefore of two pointings on each station involved, one in each position of the telescope, together with the corresponding micrometer readings, twenty-four in all, both a forward and a backward reading of each micrometer being made in each of its positions. Sixteen such series of observations are taken upon each station, one in each of sixteen positions of the horizontal circle.

As implied in this statement of the method, the instrument used carries an accurately divided horizontal circle which is read by micrometer microscopes. The circle may be shifted to different positions in azimuth, but is not provided with such a clamp and slow motion tangent screw controlling the position of the graduated circle as is needed on any instrument used with the method of repetitions.

In the repetition method of angles, one angle, between two

stations, is measured at a time. In the direction method, as practiced in the Coast and Geodetic Survey, a single series of observations serves to measure all the angles between stations, or to determine the relative directions to all stations, observed upon in that series. In the repetition method, the unknowns which are being measured, are the angles; and, in the direction method described, the unknowns are relative directions, and the difference of any two such unknowns is an angle. As this difference exists in the method of observation, it also exists in the method of adjustment. In the direction method of adjustment, the unknowns are directions, one for each line observed over from each station, and angles appear only as differences of directions.

176. In the direction method of adjustment, it is assumed that there is an error inherent in each direction observed which affects every angle involving that direction. For errors which are due to the instrument, and also those due to the observer (with one exception, noted below), there is no sufficient reason for assuming that errors are inherent in the directions, rather than in the angles. All errors due to external conditions, that is, to conditions outside the instrument and the observer, on the other hand, must be assumed to be inherent in the separate directions observed, rather than in the angles. The external errors may be due to various causes, including phase and asymmetry of the object pointed upon, eccentricity of the signals pointed upon, or of the instrument, and lateral refraction. Such errors tend to recur with one sign for each observation over a given line from a station, and therefore to affect that direction in a constant manner which is independent of the other directions which happen to enter that series of observations. In this class of external, constant errors must also be placed, whatever tendency the observer may have to misjudge the position of the center of the image pointed upon. If an observer makes every pointing on every object too far to one side, say to the left, by a constant amount, this error will not

affect the final results. But any such tendency to a one-sided pointing is reasonably certain to be modified by the brightness and size, as well as by any asymmetry, of the image pointed upon. It is therefore probable that an error of bisection exists which is peculiar to each direction observed, as long as the image of the object pointed upon in that direction remains constant in appearance.

It matters little in choosing the method of adjustment, whether the above outline of the manner in which errors are inherent in particular directions is accurate in detail or not. If the great mass of evidence available indicates strongly that the principal errors in angle measurements are of the external class, and that they are inherent in separate directions rather than in angles, the direction method of adjustment should be used in preference to the angle method. That the direction method should be used, and for this reason, is the conviction to which years of critical observation have led the computers of the Coast and Geodetic Survey. The direction method of adjustment, as set forth in this chapter, is now almost exclusively used in that Survey, even when the observations have been taken by the method of repetitions.

177. In the first part of the preceding section the direction method of observation in use in the Coast and Geodetic Survey is described. The U. S. Lake Survey is at present using a method of observation with a direction instrument which differs widely in several important respects from the Coast and Geodetic Survey method. The most important difference is that the different angles are measured independently, that is, there are but two signals pointed upon in each series of observations.

The advocates of the method of independent angles with a direction instrument urge (1) that if the twist, due to the action of the sun's rays, of the tower upon which the instrument is mounted, be considered; (2) and if the influence on distinctness of vision of the use of the same focus for lines of different lengths; (3) the interruptions that may occur in the course of

a long series; and (4) the more uniform line that may always be had when the number of signals in use at any one time be small, — be also considered, the conclusion must be reached that this method will give a greater accuracy than the other.

The advocates of the method of including all the signals which are then visible in each series, believe that the second and fourth considerations are of minor importance under actual average conditions; that the third consideration is of little importance if one does not wait (and he should not) for a signal which is not showing when a pointing is desired; and that on the towers, as now built, in the Coast and Geodetic Survey, the twist is so small as to be difficult to discover, even by special observations for that purpose, extending over long intervals of time, and therefore that the first consideration is of little importance. As against the considerations set forth above bearing upon the accuracy to be obtained, these advocates call attention to the fact that in their method the pointings upon each signal are scattered over the whole of the observing period during which that signal is visible; whereas, in the method of independent angles, the observations on each signal are confined within a few short periods; and that, therefore, a greater variety of conditions are encountered, and a tendency to greater accuracy secured in the former method. The great disadvantage of the method of independent angles lies in the greater time and cost required to secure a given number of observations. As compared with the method of independent angles, the method used in the Coast and Geodetic Survey requires but three-fourths as many pointings for a given number of observations of each angle if three signals are observed in each series on an average, and but five-eighths as many if there are five observed in each series.

It has been urged in favor of the method of independent angles that it simplifies the local adjustment. This argument had considerable force as against the method formerly used in the Coast and Geodetic Survey, but not against the present

practice in that Survey, with which no local adjustment whatever is found to be necessary.

178. The Local Adjustment.—If all signals are observed in every series, the local adjustment becomes simply a process of taking means and differences. If a broken series is observed, that is, a series in which one or more of the signals are missing, because they were not visible at the particular time when needed, the Coast and Geodetic Survey observers are at present directed that, "the missing signals are to be observed later in connection with the chosen initial, or some other one, and only one, of the stations already observed in that series." The observations on the signal which is common to the two fragments of a series are used to connect these fragments, and when so connected the series is used as if it had all been observed at one time, and no local adjustment is necessary. In this process the series which are made up of joined fragments, are given a slightly greater weight than they should be. But this is of very little importance, especially as the principal errors in angle measurements are of the systematic class and do not appear until a figure adjustment is made.

If the observations are made by the method of independent angles with a direction instrument, as in the Lake Survey, the local adjustment may be made by the methods stated in articles 121, 122.

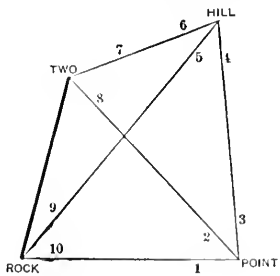

Fig. 25.

179. The Figure Adjustment.*—The following directions were observed, among others, at the stations named. It is required to adjust the quadrilateral shown in Fig. 25 by the method of directions, the line Two–Rock having been completely fixed by the adjustment of the preceding figures.

* Throughout this chapter all formulæ and examples have been given upon the assumption that all observed directions are to be assigned equal

Observed Directions.

At Two.	°	′	″	At Rock.	°	′	″
Hill	145	33	38.1	Two . . .	53	30	44.3**
Point	212	09	30.8	Hill . . .	78	36	08.7
Rock	269	41	25.2*	Point . . .	127	43	40.0

* Corrected direction, 26.3″. ** Corrected direction, 42.7″.

At Hill.	°	′	″	At Point.	°	′	″
Rock	0	00	00.0	Rock . . .	165	04	04.8
Two	30	46	43.1	Two . . .	213	19	10.7
Point	315	44	36.8	Hill . . .	251	41	04.6

The signal chosen as the initial happens to appear in these lists in only one case, Rock being the initial at Hill. There is no difference in the treatment during adjustment of the initial station and any other station.

The ten directions for which it is proposed to derive corrections are identified by numbers on Fig. 25. The convenient notation used below indicates the corrections to these figures by the same numbers inclosed in a parenthesis, thus, (1) stands for the correction to the direction numbered 1, namely, Point to Rock. No corrections are to be derived for the directions Two to Rock and Rock to Two, as these directions have already been fixed by previous adjustment.

The four condition equations are as follows, the three angle equations being given first:

Condition Equations.

$$0 = -1.3 - (1) + (2) - (8) + (10)$$
$$0 = -2.7 - (5) + (6) - (7) + (9)$$
$$0 = -7.1 - (2) + (3) - (4) + (6) - (7) + (8)$$
$$0 = -12.0 + 1.9\ (1) - 4.6\ (2) + 2.7\ (3) + 0.6\ (4)$$
$$- 3.5\ (5) + 2.9\ (6) - 4.5\ (9) + 0.6\ (10)$$

In the angle equations each required correction to an angle is expressed as a difference of two corrections to directions, except when one of the directions is already fixed, in which case the

weight. This is the case which most frequently arises. If it is desired to assign unequal weights, Chapter V shows how the weights are to be introduced.

correction to the other direction involved in the angle becomes the same (with the proper sign) as the correction to the angle.* The triangles are so small in this case that it is not necessary to take the spherical excess into account.

The side equation has its pole at Two. The numerical work of forming this side equation is shown below in a convenient form. (For an explanation of the meaning of the side equation, see Art. 126.)

Plus Terms.			
Directions.	Uncorrected Angles.	Logarithmic Sines.	Log Sine, Dif. for
	° ′ ″		1″
+ 10	74 12 57.3	9.983308	+ 0.6
− 2 + 3	38 21 53.9	9.792860	+ 2.7
− 5 + 6	30 46 43.1	9.709035	+ 3.5
	Sum	= 9.485203	
Minus Terms.			
Directions.	Uncorrected Angles.	Logarithmic Sines.	Log Sine, Dif. for
	° ′ ″		
− 1 + 2	48 15 05.9	9.872783	+ 1.9
− 4 + 6	75 02 06.3	9.985015	+ 0.6
+ 9	25 05 26.0	9.627417	+ 4.5
	Sum	= 9.485215	
	Difference . . .	= − 12	

For such directions as occur twice in the formation, as, for example, direction 2, the corresponding coefficient is the algebraic

* "The first two angle equations were purposely selected so that they refer to triangles of which the fixed line Two-Rock forms one side. Each such angle equation contains but four terms, whereas otherwise it would contain six, and the normal equations have two side coefficients which are zero. The work of forming and solving the normal equations is therefore reduced by this selection."

sum of two log sine differences for 1″, each taken with the sign fixed as indicated in the first and fourth or fifth and eighth columns of the above form, together with the headings, "plus terms" and "minus terms." The log sine differences for 1″ are given in units of sixth decimal place.

This triangulation is of tertiary character, and hence the sides are computed to six decimal places only in the logarithms. The correlate equations are shown below in the form indicated in equations (4) of Art. 110.

Correlate Equations.

$$
\begin{array}{llllcl}
-C_1 & & & +1.9\,C_4 & = & (1) \\
+C_1 & & -C_3 & -4.6\,C_4 & = & (2) \\
 & & +C_3 & +2.7\,C_4 & = & (3) \\
 & & -C_3 & +0.6\,C_4 & = & (4) \\
 & -C_2 & & -3.5\,C_4 & = & (5) \\
 & +C_2 & +C_3 & +2.9\,C_4 & = & (6) \\
 & -C_2 & -C_3 & & = & (7) \\
-C_1 & & +C_3 & & = & (8) \\
 & +C_2 & & -4.5\,C_4 & = & (9) \\
+C_1 & & & +0.6\,C_4 & = & (10)
\end{array}
$$

Note that the coefficients in the first column of these correlate equations are the same as the coefficients in the first line of the condition equations, and that the second column and the second line correspond, and so on.

The normal equations are shown below, formed from the correlate equations and condition equations as indicated in equations (5) of Art. 119.

Normal Equations.

$$
\begin{array}{lllllll}
0 = & -\ \ 1.3'' & +4.0\,C_1 & & -2.0\,C_3 & -\ \ 5.9\,C_4 \\
0 = & -\ \ 2.7'' & & +4\ \ C_2 & +2.0\,C_3 & +\ \ 1.9\,C_4 \\
0 = & -\ \ 7.1'' & -2.0\,C_1 & +2.0\,C_2 & +6.0\,C_3 & +\ \ 9.6\,C_4 \\
0 = & -12.0'' & -5.9\,C_1 & +1.9\,C_2 & +9.6\,C_3 & +73.7\,C_4
\end{array}
$$

The solution of these equations gives:

$$C_1 = +\ 1.173''$$
$$C_2 = -\ 0.110''$$
$$C_3 = +\ 1.511''$$
$$C_4 = +\ 0.063$$

These values substituted in the correlate equations give the required corrections:

$$
\begin{array}{llll}
(1) & = -\ 1.05'' & (6) & = +\ 1.58'' \\
(2) & = -\ 0.63'' & (7) & = -\ 1.40'' \\
(3) & = +\ 1.68'' & (8) & = +\ 0.34'' \\
(4) & = -\ 1.47'' & (9) & = -\ 0.39'' \\
(5) & = -\ 0.11'' & (10) & = +\ 1.21''
\end{array}
$$

After applying these corrections to the directions, the computation of triangle sides shows every triangle with the sum of its corrected angles equal to $180°\ 00'\ 00.0''$, there being no spherical excess in these small triangles, and shows the lengths of the lines to be the same when computed in two possible ways through the triangles. The correctness of the adjustment is thus checked.

The procedure in adjusting any figure, however complicated, in which the only thing fixed by previous adjustment is one line, is not essentially different in any respect from that here illustrated by the simple case of a quadrilateral with one fixed line.

The number of angle and side condition equations in any particular figure is to be determined as indicated in Arts. 144, 151, 152. For examples of condition equations for complicated figures, see Appendix 4 of the Coast and Geodetic Survey Report for 1903.

180. The Best Side Equations.—In Arts. 153, 154, certain suggestions are given as to the manner in which side equations should be selected to avoid the danger that the solution of the normal equations may be an unstable one, that is, a solution in which the effect of omitted decimal places on the derived values of the required unknowns is large, and in which it is, therefore,

necessary to carry a large number of decimal places in the solution to secure the unknowns with certainty to a small number of decimal places. The suggestions are difficult, but important, to follow. This article will serve to illustrate these suggestions by a concrete case,* namely, that shown in Fig. 25,

The observed directions are as follows:

At Spear.	°	′	″	At Tobacco Row.	°	′	″
Long . . .	0	00	00.000	Spear . . .	0	00	00.000
Smith . . .	6	04	57.749	Long . . .	72	37	08.593
Flat Top . .	37	00	48.900	Smith . . .	118	11	11.341
Tobacco Row,	47	03	16.925	Flat Top . .	159	40	31.200

At Long.	°		
Smith	0	00	00.000
Flat Top	57	52	28.128
Tobacco Row	108	47	15.636
Spear	169	06	53.169

At Smith.	°	′		At Flat Top.	°	′	″
Flat Top . .	0	00	00.000	Tobacco Row,	0	00	00.000
Tobacco Row,	30	12	41.103	Spear . . .	10	17	00.258
Spear . . .	51	03	16.151	Long . . .	42	01	51.794
Long . . .	55	51	27.879	Smith . . .	108	18	02.385

The figure requires three side equations. Let each side equation be represented symbolically by the abbreviation inclosed in a parenthesis for the station used as a pole, followed in order by the abbreviations for each of the other stations at the ends of the sides involved. As in Art. 179, the required correction to a direction will be indicated by the number of that direction inclosed in a parenthesis. The numbers assigned to the directions are indicated on the figure. The side equations represented by the symbols (L.)–Sm.–T. R.–Sp., (L.)–Sm.–F. T.–Sp., and (L.)–Sm.–F. T.–T. R. are

* This article, as well as the suggestions in Art. 154, is based on pp. 118–120 of the C. and G. S. Rep. for 1878 (App. No. 8) written by Mr. M. H. Doolittle.

$$0 = +1.2'' - 17.80\,(1) + 19.76\,(2) - 1.96\,(4) - 0.66\,(5) + 2.72\,(6) - 2.06\,(7) - 4.38\,(14) + 25.05\,(15) - 20.67\,(16). \quad (1)$$

$$0 = -1.9'' - 16.97\,(1) + 19.76\,(2) - 2.79\,(3) - 1.43\,(13) + 25.05\,(15) - 23.62\,(16) - 3.40\,(18) + 4.33\,(19) - 0.93\,(20). \quad (2)$$

$$0 = -1.9'' - 1.95\,(6) + 2.06\,(7) - 0.11\,(8) - 1.43\,(13) + 4.38\,(14) - 2.95\,(16) - 2.33\,(17) + 3.26\,(19) - 0.93\,(20). \quad (3)$$

The directions 1, 2, 15, and 16 are the sides of the small angles at Spear and Smith. In the first two equations the coefficients of the corrections (1), (2), (15), and (16) so largely predominate over everything else, and for corresponding terms are so nearly equal, that the two equations may be considered approximately identical. The effect of this would be to make certain side coefficients in the normal equations about as large as the corresponding diagonal coefficients, and the solution would be unstable. An attempt at solution with factors extending to but three significant figures would not be likely to furnish even an approximation to the values of the unknown quantities.

Adding the third equation to the first, and subtracting the second from the sum, the following equation results:

$$0 = +1.2'' - 0.83\,(1) + 2.79\,(3) - 1.96\,(4) - 0.66\,(5) + 0.77\,(6) - 0.11\,(8) - 2.33\,(17) + 3.40\,(18) - 1.07\,(19). \quad (4)$$

It is evident that if the first, third, and fourth equations are satisfied, the second must also be satisfied. Hence, the fourth equation may be safely substituted for the second, or, as will easily appear, for the first, if it be preferred to retain the second.

The fourth equation corresponds to the symbol (L.)–F. T.–T. R.–Sp., and might have been obtained directly in the usual way. The second suggestion in Art. 154, to use small angles once, and only once, would have led to the selection of the fourth equation in the place of the second.

The second suggestion of Art. 154 may be carried out still more fully by so selecting one of the side equations as to involve the small angles of the triangle Sp.–T. R.–F. T. Let the side equation, represented by the symbol (T. R.) – Sp. – L.– F. T., be used. It is:

$$0 = -4.0'' + 1.96\,(1) - 11.89\,(3) + 9.93\,(4) + 1.71\,(10) \\ -2.91\,(11) + 1.20\,(12) + 9.27\,(17) - 11.60\,(18) + 2.33\,(19). \quad (5)$$

To carry out the third suggestion of Art. 154, that it is sometimes desirable to use a side equation of large scope, let (L.)–Sm.–F. T.–Sp.–T. R. be used. It is:

$$0 = +3.1'' - 0.83\,(1) + 2.79\,(3) - 1.96\,(4) - 0.66\,(5) + 2.72\,(6) \\ - 2.06\,(7) + 1.43\,(13) - 4.38\,(14) + 2.95\,(16) \\ + 3.40\,(18) - 4.33\,(19) + 0.93\,(20). \quad (6)$$

The three side equations recommended as best are, then, the sixth, third, and fifth.

It is interesting to note that the sixth equation is the same as the first minus the second.

181. Length, Azimuth, Latitude and Longitude Condition Equations. — In the adjustment of a triangulation, it has been shown thus far how in a net joining several stations the conditions arising from the closure of triangles and from the equality of lengths or sides as computed by different routes can be satisfied. Cases frequently arise in which other condition equations must be satisfied, in addition to the angle and side equations. These will now be treated briefly.

The principal cases which arise are the following three, and various combinations of them:

1. A section of triangulation which starts from a line which is fixed in length, either by direct measurement or by previous triangulation, may end on a line which is similarly fixed in length. In this case a length condition equation must be used to make the length of one of these fixed lines as computed

through the adjusted triangulation from the other fixed line agree with its fixed value.

2. A section of triangulation may include two lines which are each fixed in azimuth, either by previous triangulation, or by astronomic observations, which in the particular case furnish a determination of the azimuth stronger than that given by the triangulation. It is then necessary to use an azimuth condition equation to insure that the azimuth as carried by computation through the adjusted triangulation from one fixed line shall agree at the other fixed line with the azimuth as already fixed there.

3. The latitude and longitude of each of the two stations in the section of triangulation may have been fixed by previous triangulation, and it may be desired to retain these positions unchanged. This may be done by writing a latitude condition equation, and a longitude condition equation, such as to insure that the latitude and longitude as computed from the first fixed station through the adjusted triangulation shall agree at the second fixed station with the fixed latitude and longitude there.

As a belt of triangulation is gradually extended, it frequently happens that it returns upon itself in such a manner as to form a complete circuit. In such a case it usually happens that before the circuit is closed by the field operations a considerable portion of the triangulation in the circuit has been adjusted, the results used in various ways, and perhaps published. It is then desired to adjust the last portion of the circuit, that is, the portion still unadjusted when the last of the field work is done, so that no discrepancies of any kind remain at the closing line. In this case it is necessary to use, in this last portion, in addition to the angle and side equations, one condition equation of each of the other kinds.

In any case, when the necessary condition equations referring to length, azimuth, latitude and longitude have been formed, they may be placed with the angle and side condition equations, and the formation of correlates and of normal equations and the

solution of the normal equations may be made as indicated in Art. 179.

182. Length Condition Equations. — A length condition equation is written in the same form as a side equation, and is essentially of the same character. It serves to insure that the length of the second of the two lines which are fixed in length by direct measurement, or by other triangulation, shall, as computed through the adjusted triangulation from the first fixed line, agree with its fixed value. The form of the equation is, when expressed in terms of angles:

$$0 = (\log a) + \delta_{A_1}(A_1)\ \delta_{B_1}(B_1) + \delta_{A_2}(A_2)\ \delta_{B_2}(B_2) + \delta_{A_3}(A_3)\ \delta_{B_3}(B_3), \ldots$$

in which, in a selected chain of triangles from the first to the second fixed line, A_1, A_2, A_3 . . . is in each case the angle opposite the required side, and B_1, B_2, B_3 . . . is the angle opposite the known side.* (A_1), (A_2), (A_3) . . ., (B_1), (B_2), (B_3) . . ., are the required corrections to these angles. δ_{A_1}, δ_{A_2}, δ_{A_3} . . ., δ_{B_1}, δ_{B_2}, δ_{B_3} . . ., are the differences for 1″ in the logarithmic sines of the angles A_1, A_2, A_3 . . ., B_1, B_2, B_3 . . . . $(\log a)$ is the necessary correction to the logarithm of the second fixed side, a, as computed through the selected chain of triangles from the first fixed side, b, by the law of the proportion of sines, using the uncorrected angles A_1, A_2, A_3 . . ., B_1, B_2, B_3 . . ., to make it agree with the fixed value of log b.

The proof that the formula given above is a proper expression of the length condition, is similar to that given in Art. 147, for the form of the side condition equation there derived.

To express this equation in terms of corrections to directions, all that is necessary is to substitute in each case for (A_1), (A_2), (A_3) . . ., (B_1), (B_2), (B_3) . . ., which are corrections to angles, the corresponding differences of two corrections to directions, if neither side of the angle is fixed, or one correction to a direc-

* These angles A_1, A_2, A_3 . . ., B_1, B_2, B_3 . . ., may appropriately be called distance angles. Their locations in a chain of triangles are illustrated in Fig. 25.

tion with the proper sign, if one side of the angle is a line fixed in direction.

Any chain of triangles between the fixed lines may be selected. It is well to select a chain containing a minimum number of triangles, thereby making the number of terms in the condition equations, as small as possible, and reducing the labor of solution. It is well, also, in order to avoid instability in the solution of the normal equations, to select the strongest possible short chain, that is, a chain in which small angles are avoided as far as possible.

The number of independent length condition equations in any figure is one less than the number of lines which are fixed in length. As, for example, if there are three fixed lengths in a figure, and length condition equations are written fixing the ratios of the first and second, and first and third, of the fixed lines as computed through the adjusted triangulation, the ratio of the computed lengths of the second and third lines is thereby fixed. Any third length condition equations will therefore be derivable from the condition equations already written, not independent of them.

The rigorous adjustment is made by adding the length condition equation, or equations, to the other condition equations, proceeding with the formation of correlates and normal equations as indicated in Art. 179. Various approximate methods of adjustment have been proposed and used in the place of the rigorous adjustment. The experience of the Coast and Geodetic Survey indicates that it is seldom advisable to use any of these approximate solutions, other than those of the nature indicated in Art. 186.

183. Azimuth Condition Equations. — An azimuth condition equation serves to insure that the azimuth of the second of the two lines which are fixed in azimuth, by observations external to the section of triangulation being adjusted, shall, as computed through the adjusted triangulation from the first fixed line, agree with its fixed value.

Let the selected chain of triangles, for a section of triangulation used in writing the length condition equation as indicated in the preceding article, be illustrated by Fig. 26. Let it be supposed that the order of computation is that indicated by the numbering of the triangles 1 to 6. The distance angles used in forming the length condition equation are marked by the letters A and B. Let it be supposed that the lines $D\dot{E}$ and NO are fixed in azimuth.

For the same reason that in forming the length condition equations the number of terms was kept as small as possible, namely, to save work in the computation, so here the azimuth condition equation should directly involve as few angles as possible. It is possible to carry the azimuth by computation through a chain of triangles by using only one angle in each triangle (the lengths being supposed known), namely, the angle marked C in Fig. 26. This third angle in each triangle will, for convenience, be called the azimuth angle to distinguish it from the distance angles. The azimuth condition equation should involve the azimuth angles only in the selected chain of triangles through the figure used in writing the length equation.

As a given correction to any of the azimuth angles in Fig. 26 affects the azimuth of the line NO as computed through the adjusted triangulation by precisely the amount of correction to that angle, it is evident that the general form of the azimuth condition equation is

$$0 = -(a) + \Sigma(C_R) - \Sigma(C_L),$$

in which (a) is the correction to the azimuth of the second fixed line, as computed through the unadjusted azimuth angles from the first fixed line, necessary to make it agree with the fixed azimuth there, and $\Sigma\ C_R)$ is the sum of the corrections to the azimuth angles which are on the right-hand side of the chain of triangles when proceeding in the direction of computation, and $\Sigma(C_L)$ is the sum of corrections to the left-hand azimuth angles.

For example, in Fig. 26 the azimuth condition equation is

$$0 = -(a) + (C_1) - (C_2) - (C_3) + (C_4) + (C_5) + (C_6),$$

in which (a) is the fixed azimuth of NO minus its azimuth as computed through the unadjusted angles C_1, C_2, C_3, . . . C_6 from the fixed azimuth of DE.

To express this condition equation in terms of directions, it is necessary simply to substitute for each correction to an angle the difference of the two corrections to directions, one of which is necessarily zero if that particular direction is already fixed.

As the number of length condition equations is one less than the number of fixed lengths in a figure, so the number of azimuth condition equations is one less than the number of fixed azimuths.

184. Latitude and Longitude Condition Equations.—A latitude condition equation serves to insure that the latitude of the second of the two points which are fixed in latitude, by observations external to the triangulation being adjusted, shall, as computed through the adjusted triangulation from the first of two points fixed in latitude, agree with its fixed value. A longitude condition equation serves the same purpose with respect to longitudes.

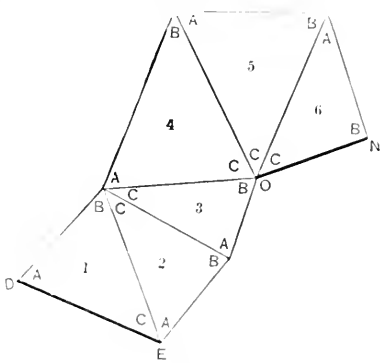

Fig. 26.

In Fig. 26 the latitude and longitude of each of the points E and N are supposed to be fixed. It is supposed that the position $\phi_n\lambda_n$ of point N as computed through the unadjusted triangulation differs from its fixed position $\phi_{n'}\lambda_{n'}$. It is required to write the latitude and longitude condition equations necessary to remove the discrepancy.

In the figure the azimuth angles have been marked C, and the distance angles, of two classes, are marked A and B, the A

angle being in each case opposite the required side, and the B angle being opposite the known side. The computation is supposed to proceed from DE to ON.

The azimuth is supposed to have been computed through the angles C.

Let δ_A, δ_B be the logarithmic sine differences for $1''$ for an A distance angle and a B distance angle.

Let M be the modulus of the common system of logarithms.

Let $$r_1 = \frac{B_c}{A_n} \cos \phi_n \text{ arc } 1'' \text{ and } r_2 = \frac{A_n}{B_c} \sec \phi_n \text{ arc } 1''.$$

In the expressions for r_1 and r_2, A_n is the value at the point N of

$$A = \frac{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}{a \text{ arc } 1''},$$

and B_c is the value at the vertex of any C angle of

$$B = \frac{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}{a\,(1 - e^2) \text{ arc } 1''}.$$

In these expressions for A and B, a is the equilateral radius of the spheroid on which the triangulation is computed, and e is its eccentricity. A and B are factors used in the computation geodetic positions. For their values on the Clark Spheroid of 1866, see Appendix 9 of the Coast and Geodetic Survey Report for 1894.

In each triangle, reckoning all the angles in seconds, a correction (A) in a distance angle A will produce a correction in the computed latitude at N of $\dfrac{(\phi_n - \phi_c)\,\delta_A\,(A)}{M}$, and a correction in a computed longitude of $\dfrac{(\lambda_n - \lambda_c)\,\delta_A\,(A)}{M}$. A correction to a distance angle B produces similar corrections to the latitude and longitude at N, but with the reverse algebraic sign.

$\phi_c\lambda_c$ is the position of the vertex of the C angle in the triangle containing the angle which is supposed to be corrected.

Let any azimuth angle on the right side of the chain of triangles proceeding in the direction of progress of computation

be designated by C_R, and any such angle on the left side by C_L. Then any correction to a right azimuth angle produces a correction in latitude at N of $+ r_1 (\lambda_n - \lambda_c) (C_R)$ and in longitude of $- r_2 (\phi_n - \phi_c) (C_R)$. Similarly, any correction to a left azimuth angle produces the corrections $- r_1 (\lambda_n - \lambda_c) (C_L)$ and $+ r_2 (\phi_n - \phi_c) (C_L)$ to the latitude and longitude respectively at N.* Hence,

$$0 = \phi_n - \phi_n' + \Sigma\left[\frac{(\phi_n - \phi_c)\, \delta_A\, (A)}{M} - \frac{(\phi_n - \phi_c)\, \delta_B\, (B)}{M} \right.$$
$$\left. + r_1 (\lambda_n - \lambda_c) (C_R) - r_1 (\lambda_n - \lambda_c) (C_L)\right]. \tag{1}$$

$$0 = \lambda_n - \lambda_{n'} + \Sigma\left[\frac{(\lambda_n - \lambda_c)\, \delta_A\, (A)}{M} - \frac{(\lambda_n - \lambda_c)\, \delta_B\, (B)}{M} \right.$$
$$\left. - r_2 (\phi_n - \phi_c) (C_R) + r_2 (\phi_n - \phi_c) (C_L)\right]. \tag{2}$$

With sufficient accuracy and greatest convenience $\phi_n - \phi_c$ and $\lambda_n - \lambda_c$ may usually be reckoned in minutes and tenths. Then, multiplying the equations by $\frac{M}{60} = \left(\frac{.4343}{60}\right)$ in order to remove a constant factor from the coefficients of (A) and (B),

$$0 = .00724\, (\phi_n - \phi_n') + \Sigma[(\phi_n - \phi_c)\, \delta_A (A) - (\phi_n - \phi_c)\, \delta_B (B)$$
$$+ .4343\, r_1 (\lambda_n - \lambda_c) (C_R) - .4343\, (\lambda_n - \lambda_c) (C_L)]. \tag{3}$$
$$0 = .00724\, (\lambda_n - \lambda_{n'}) + \Sigma[(\lambda_n - \lambda_c)\, \delta_A (A) - (\lambda_n - \lambda_c)\, \delta_B (B)$$
$$- .4343\, r_2 (\phi_n - \phi_c) (C_R) + .4343\, r_2 (\phi_n - \phi_c) (C_L)]. \tag{4}$$

185. If these condition equations are used in this form, and if there is a length condition equation and an azimuth condition equation covering the same chain of triangulation, it will be found that the normal equations have large side coefficients, and their solution will be unstable. The following transformation facilitates the solution by avoiding this difficulty.

* To show the derivation of the formulæ for r_1 and r_2 would require considerable space. This derivation has no bearing upon the method of least squares. It suffices for the present purpose to note that r_1 and r_2 are functions of the latitude, and that the formulæ given show the relation between (C) and the corresponding corrections in latitude and longitude at N.

Assume a point and $\phi_h\lambda_h$ approximately in the mean latitude and longitude of the points $\phi_c\lambda_c$ which are at the vertices of the C angles. Multiply the length condition equation given in Art. 152, which is in the form

$$0 = (\log a) + \Sigma\, \delta_A\, (A)\ \Sigma\, \delta_B\, (B), \text{ by } \phi_h - \phi_n,$$

and the azimuth condition equation given in Art. 153, namely,

$$0 = -\ (a) + \Sigma\, (C_R) - \Sigma\, (C_L), \text{ by } .4343\, r_1\, (\lambda_h - \lambda_n),$$

and add the products to equation (3). Also multiply the length equation by $(\lambda_h - \lambda_n)$ and the azimuth equation by $-\ .4343\ r_2$ $(\phi_h - \phi_n)$ and add the products to equation (4). Then the latitude and longitude condition equations are as follows, in which all that portion within the first square bracket in each case constitutes the absolute term:

$$\begin{aligned} 0 = [&.00724\, (\phi_n - \phi_{n'})\, (\phi_h - \phi_n)\, (\log a) + .4343\, r_1\, (\lambda_h - \lambda_n)\, (a)] \\ &+ \Sigma\, [(\phi_h - \phi_c)\, \delta_A\, (A) - (\phi_h - \phi_c)\, \delta_B\, (B) \\ &+ .4343\, r_1\, (\lambda_h - \lambda_c)\, (C_R) - .4343\, (\lambda_h - \lambda_c)\, (C_L)]. \end{aligned} \tag{5}$$

$$\begin{aligned} 0 = [&.00724\, (\lambda_n - \lambda_n')\, (\lambda_h - \lambda_n)\, (\log a) - .4343\, r_2\, (\phi_h - \phi_n)\, (a)] \\ &+ \Sigma\, [(\lambda_h - \lambda_c)\, \delta_A\, (A) - (\lambda_h - \lambda_c)\, \delta_B\, (B) \\ &- .4343\, r_2\, (\phi_h - \phi_c)\, (C_R) + .4343\, r_2\, (\phi_h - \phi_c)\, (C_L)]. \end{aligned} \tag{6}$$

This transformation is allowable since the adjusted angles which satisfy the length and azimuth equations and equations (5) and (6) must evidently satisfy (3) and (4) also, and no new condition has been put into the solution, as (5) and (6) are mere combinations of the original conditions.

The transformation has the effect of transferring the point at which the discrepancy in latitude and longitude is supposed to have developed from the end of the triangulation $\phi_n\lambda_n$ to a mean point $\phi_h\lambda_h$. Equations (3) and (4) correspond to the supposition that the computation of latitudes and longitudes has progressed continuously from the beginning of the section of triangulation being adjusted to $\phi_n\lambda_n$. Equations (5) and (6) correspond to the supposition that the computation of latitudes and longitudes has been made in two sections, one from the

beginning of the section being adjusted to $\phi_h\lambda_h$: and the other from the end of the section (at $\phi_n\lambda_n$) back to $\phi_h\lambda_h$, this portion of the position computation being supposed to be made by starting with the fixed length and azimuth found at a line of which $\phi_n\lambda_n$ is one end; and that the discrepancy in latitude and longitude is developed at $\phi_h\lambda_h$ by the junction of the two position computations there.

To express the latitude and longitude condition equations as written above in terms of corrections to directions, it is necessary simply to substitute for each correction to an angle the difference between two corrections to directions, one of which is necessarily zero if that particular direction is fixed.

The number of latitude condition equations in a figure is one less than the number of groups of points fixed in latitude, each group being composed of points which are tied together by lines already fixed in length and azimuth, and being separated from other groups by lines not so fixed. The removal of the latitude discrepancy for one point of such a group removes it for all. Similar statements are true for the longitude condition equations.

On the Breaking of a Net of Triangulation into Sections for Convenience of Solution.

186. In a long chain of triangulation, or in a complicated net, the simultaneous solution of the condition equations, which are required by theory to secure the ideal most probable results, would be very troublesome, not from any principle involved, but from its very unwieldiness. For example, such a simultaneous solution for all the primary triangulation now forming a continuous net in the United States would involve more than two thousand condition equations. Accordingly it is necessary to break the triangulation into sections and adjust each section by itself. As this breaking into sections causes more or less disturbance of the ideal solution at or near the lines of breaking, the exercise of judgment is required in the selection of these

lines. The larger the sections are made, the nearer the approximation to the ideal solution, and on the other hand the greater will be the labor of computation.

The present practice in the Coast and Geodetic Survey is to divide the triangulation into sections, adjust each section by the rigorous method set forth in this chapter, and, in commencing the adjustment of each new section, to hold as absolutely fixed in all respects all lines which have entered into a previous adjustment. This method has the great advantage of giving the final results by a single adjustment. It is believed that with a proper selection of lines of separation between adjustments the results obtained are so close an approximation to the ideal best results that no further expenditure of energy in computation is warranted. Two principles guide in selecting the lines of separation. First, such a line should be one which is strongly determined. Second, it should be a line which enters in but few conditions in one of the two sections. For example, a line at the edge of a base net and forming the first line of the chain of triangulation connecting that base net with the next is frequently selected as a line of separation. It is strongly determined in length by the base and base net. It is usually involved in but few conditions on the side toward the chain of triangulation. In the separation into sections in primary triangulation the most frequent cases are:

1. A section which begins with a line at the margin of one base net and extends to a line at the far side of the next base net, thus including the second base net.

2. A section which includes two base nets and the triangulation between them.

3. A section which is limited at each end by a line fixed by previous adjustment in azimuth, latitude, and longitude as well as in length.

In secondary and tertiary triangulation a great variety of cases arise, and, in general, the adjustment is made in smaller sections than the primary triangulation.

CHAPTER VIII

APPLICATION TO BASE-LINE MEASUREMENT AND TO LEVELING

187. Precision of a Base-Line Measurement.—For clearness it will be necessary to outline the principles on which the measurement is made.

First, we must find the length of the measuring bar or tape in terms of some standard of length; and as the measurements of the line itself are made at various temperatures, the coefficients of expansion of the metals in the measuring apparatus must also be known. Comparisons must, therefore, be made with the standard during wide ranges of temperature; and as these comparisons are fallible, the results found for length and expansion will be more or less erroneous.

The principle involved in the measurement is exactly the same as in common chaining with chain and pins. There are, indeed, various contrivances for getting a precision not looked for in chaining, such as for aligning the measuring bar, for finding the inclination of each position of the bar, and for establishing fixed points for stopping at and starting from in measurement. But these make no change in the essential principle.

The errors in the value of a base line may, therefore, be considered to arise from two principal sources,—comparisons and measurement. Experience has shown that a considerable portion of the error in the length of a base arises from the errors in the comparisons which serve to determine the length of the measuring bar or tape.

These errors differ essentially in character. An error arising from the comparisons, being the same for each bar measurement, is cumulative for the whole base, while errors arising in the measurement of the base itself, were the measurements

repeated often enough and the conditions sufficiently varied, would tend to mutually balance, and could, therefore, be treated by the strict principles of least squares. But as the number of measurements is not often more than 2 or 3, and as these are made usually at about the same season of the year, only a comparatively rough estimate of the precision is to be looked for.

As a check on the field work a base is usually divided into sections by setting stones firmly in the ground at approximately equal intervals along the line, so that instead of being able to compare results at the end points only, we may compare results just as well at 6 or 8 points. In this way a better idea of the precision of the work is obtained, as we have 6 or 8 short bases to deal with instead of a single long one.

We proceed now with the problem of determining the precision of measurement. It may be stated as follows: A base is measured in n sections with a bar of a certain length, each section being measured n_1 times. By the first measurement the first section contains M_1' bars, the second M_1'' bars, . . . ; by the second measurement the first section contains M_2' bars, the second M_2'' bars, . . . ; and so on. The weights of the measurements in order being p_1', p_2', . . . ; p_1'', p_2'', . . . ; . . . respectively, required the m. s. e. of the most probable value of the base.

Let x_1 = most prob. value of first section,
x_2 = most prob. value of second section,
. . . .

then we have the observation equations:

First section,
$$x_1 - M_1' = v_1' \quad \text{wt. } p_1',$$
$$x_1 - M_1'' = v_1'' \quad \text{wt. } p_1'',$$
. . . .

Second section,
$$x_2 - M_2' = v_2' \quad \text{wt. } p_2',$$
$$x_2 - M_2'' = v_2'' \quad \text{wt. } p_2'',$$
. . . .

and so on.

Now, either of two assumptions may be made.

188. (*a*) In the first place, that the precision of the measurement of each bar length is the same throughout the different sections.

We have, then, nn_1 equations containing n unknowns, and the normal equations are

$$\begin{aligned} +[p_1]x_1 \qquad\qquad &= [p_1M_1], \\ +[p_2]x_2 &= [p_2M_2], \\ \cdots\cdots\cdots & \end{aligned}$$

whence $x_1, x_2, \ldots$ are known, and therefore the whole line $x = x_1 + x_2 + \ldots + x_n$ is known.

The probable error r of an observation of weight unity — that is, of a single measurement of a bar length — is given by (see Art. 105)

$$r = 0.6745 \sqrt{\frac{[pv^2]}{\text{No. of obs.} - \text{No. of indep. unknowns}}}$$

$$= 0.6745 \sqrt{\frac{[pv^2]}{n(n_1 - 1)}}.$$

Now, the length of the measuring bar being taken as the unit of measurement, the weight of a section, as depending on the measurement, may be expressed in terms of the number of bar lengths measured. For since r is the p. e. of a measurement of a single bar length, the p. e. of the measurement of a length of M bars is $r\sqrt{M}$. Hence $\frac{1}{M}$ is the weight of a measurement of length M when the weight of a measurement of the unit of length is unity.

Writing, therefore, for the weights p their values in terms of M,

$$r = 0.6745 \sqrt{\frac{1}{n(n_1 - 1)}\left[\frac{vv}{M}\right]}.$$

In the case usually occurring in practice, where the line is measured twice, we may put this formula in a form more convenient for computation. For if the first measurement of the

n_1 sections gave lengths M_1, M_2, . . . , and the second measurements gave lengths $M_1 + d_1$, $M_2 + d_2$, . . . for the same sections in order, then, since

$$[v^2] = \left[\left(+\frac{d}{2}\right)^2 + \left(-\frac{d}{2}\right)^2\right] = \frac{[d^2]}{2},$$

we have for the m. s. e. of one measurement of a bar length and for the mean of two measurements respectively,

$$\sqrt{\frac{1}{2n}\left[\frac{d^2}{M}\right]} \text{ and } \frac{1}{2}\sqrt{\frac{1}{n}\left[\frac{d^2}{M}\right]}.$$

Hence the p. e. of a single measurement and of the mean of the two values of the whole base are

$$0.67\sqrt{\frac{[M]}{2n}\left[\frac{d^2}{M}\right]} \text{ and } 0.34\sqrt{\frac{[M]}{n}\left[\frac{d^2}{M}\right]},$$

the number of bar lengths in the line being $[M]$.

Ex. — The Bonn Base, measured in 1847, near Bonn, Germany, with the original Bessel metallic-thermometer apparatus. The base was a broken one, the two parts making an angle of 179° 23′. Each part was measured twice as follows: *

	Differences.	No. of Bar Lengths.	
Northern Part	L		
Sec. 1	− 0.183	116	
Sec. 2	+ 0.094	87	
Sec. 3	− 0.013	61	264
Southern Part			
Sec. 1	− 0.007	92	
Sec. 2	+ 0.095	60	
Sec. 3	+ 0.757	131	283

Hence the m. s. e. of the northern part, arising from errors of measurement only, is

$$\frac{1}{2}\sqrt{\frac{264}{3}\left(\frac{.183^2}{116} + \frac{.094^2}{87} + \frac{.013^2}{61}\right)} = \pm \overset{L}{0.093},$$

* *Das rheinische Dreiecksnetz.* Berlin, 1876.

and the m. s. e. of the southern part is

$$\frac{1}{2}\sqrt{\frac{283}{3}\left(\frac{.007^2}{92}+\frac{.095^2}{60}+\frac{.757^2}{131}\right)}=\pm\overset{L}{0.327}.$$

The other two main sources of error are:

1. Error in comparison of the measuring bars with one another.
2. Error in the determination of their length.

The m. s. e. arising from these sources are, respectively,

$$\pm\overset{L}{0.386},\quad \pm\overset{L}{0.313} \text{ for the northern part,}$$
$$\pm 0.391,\quad \pm 0.335 \text{ for the southern part.}$$

Remembering that these latter errors are systematic, we have, finally,

$$\begin{aligned}\text{p. e. of base} &= .6745\sqrt{.093^2+.327^2+(.386+.391)^2+(.313+.335)^2}\\ &= \overset{L}{0.72}.\end{aligned}$$

189. (*b*) In the second place, if we assume that the law of precision of the measurements of the different sections is unknown, and that these sections are independent, we have for the mean of the values of the several sections and their m. s. e.,

$$x_1=\frac{[p_1M_1]}{[p_1]},$$

$$\mu_{x_1}^2=\frac{[p_1v_1^2]}{[p_1](n_1-1)}=\frac{[v_1^2]}{n_1(n_1-1)}\quad\text{since } p_1'=p_1''=\cdots=\frac{1}{M_1},$$

$$x_2=\frac{[p_2M_2]}{[p_2]},$$

$$\mu_{x_2}^2=\frac{[p_2v_2^2]}{[p_2](n_1-1)}=\frac{[v_2^2]}{n_1(n_1-1)}\quad\text{since } p_2'=p_2''=\cdots=\frac{1}{M_2},$$

. . . .

If x denotes the whole line, so that

$$x=x_1+x_2+\cdots+x_n,$$

then, since the measurements are independent,

$$\begin{aligned}\mu_n^2&=\mu_{n_1}^2+\mu_{n_2}^2+\cdots\\ &=\frac{1}{n_1(n_1-1)}\left([v_1^2]+[v_2^2]+\cdots\right)\end{aligned}$$

and the (m. s. e.)^2 of a single measurement of the line

$$\frac{1}{n_1 - 1}([v'^2_1] + [v'^2_2] + \cdots).$$

The number of bar lengths in the line being $[M]$, we have for the average value of the (m. s. e.)^2 of a single measurement of a bar length

$$\frac{1}{n_1 - 1}\,\frac{[v'^2_1] + [v'^2_2] + \cdots}{[M]}.$$

If, for example, the line has been measured twice, and $d_1, d_2, \ldots d_n$ are the differences of the measurements of the several sections, then

$$[v'^2_1] = \frac{d_1^2}{2},$$

$$[v'^2_2] = \frac{d_2^2}{2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and therefore

$$\mu_{l'}^2 = \frac{[d^2]}{4},$$

and the (m. s. e.)^2 of a single measurement of a bar length is $\frac{1}{2}\frac{[d^2]}{[M]}$.

Ex. — The Chicago Base, measured in 1877 with the Repsold metallic-thermometer apparatus belonging to the United States Engineers. The base was divided into 8 sections, and was measured twice.

Section.	No. of Bar Lengths.	Dif. of Measures.
		mm.
I.	227.25	− 1.3
II.	230.25	+ 2.5
III.	234.50	+ 2.3
IV.	232	+ 0.7
V.	231	+ 1.5
VI.	225	+ 1.1
VII.	300.50	+ 1.3
VIII.	196.80	− 0.2

Taking the errors of the different sections as independent, the p. e. of the mean of the two measures of the base is

$$0.6745 \sqrt{\frac{[d^2]}{4}} = \overset{mm.}{1.46}.$$

The p. e. arising from the other sources of error were

	mm.
(1) Measuring bar	$\pm$ 6.38,
(2) Metallic thermometer	$\pm$ 2.82,
(3) Elevation above mean tide, N.Y.	$\pm$ 0.36.

Assuming these to be independent, the p. e. of the Chicago Base at sea level is

$$\sqrt{1.46^2 + 6.38^2 + 2.82^2 + 0.36^2} = \overset{mm.}{7.14}.$$

APPLICATION TO LEVELING.

190. Lines of levels are usually run in duplicate, each portion being leveled over twice, sometimes both levelings being in the same direction, but preferably in opposite directions. The probable error of the mean result for a single kilometer, or for the whole line, may then be computed from the discrepancy between the two levelings over each section by the formulæ given in the preceding article, in connection with base line measurements.

Whenever a series of such lines becomes so connected as to form a network, an additional determination of the accuracy of the work is afforded by the closing errors of the various circuits forming the net. Experience shows that the probable errors as thus computed from the adjustment are usually considerably larger than as computed from the discrepancy on short sections between the two runnings of each line, thus indicating that there are systematic errors in the leveling which are not eliminated by duplicating each line.

The rigorous adjustment of a level net may be made in either of two ways, namely, by the method set forth in Chapter IV in dealing with indirect observations, or by the method of Chapter V which is applicable to conditioned observation.

191. Method of Indirect Observations. — Let it be supposed that s is the total number of junction bench-marks in a net of leveling, each of these junction bench-marks being common to three lines of the net. Let l be the number of lines in the net, each connecting two junction bench-marks. Now if it be assumed that the unknowns desired are the elevations of each of $s - 1$ of these junction bench-marks, referred to the remaining bench-mark as a zero point, the case in hand is one of indirect observations with the no conditions. There are $s - 1$ unknowns, and l observation equations each of the form

$$x - y = l \quad \text{or} \quad x = l,$$

see equations (2) of Art. 77, and the solution is carried out in accordance with Chapter IV.

192. Method of Conditioned Observations. — As before, let it be supposed that s is the total number of junction bench-marks in the net l and the number of lines each connecting two junction bench-marks. If it be assumed that the required unknowns are the l differences of elevations between the junction bench-marks at the ends of each of the l lines, the case in hand is one of conditioned observations. There will be l observation equations each of the form $x = l$ expressing the direct observation in each case of one of the required unknowns. The number of differences necessary to fix the relative elevations of s bench-marks is $s - 1$. The number of observed differences in excess of this required number is $l - (s - 1)$, and therefore the number of condition equations is $l - s + 1$. These conditions exist as the requirement that each circuit in the net must close, that is, the sum of the differences of elevation in order around each circuit must be zero. The solution may be carried out as indicated in Chapter V by the Method of Correlates (see Art. 119).

193. One question which arises at the outset with either of these two general methods of solution is, what relative weights shall be assigned to the different lines. If all the leveling is done with one type of instrument by a uniform method and

under similar conditions, the problem of assigning weights is simply that of determining the relation between the errors in the lines and their lengths. If all of the errors are of the accidental class, the accumulated errors in different lines tend to be as the square roots of their lengths, and therefore the proper weights are inversely as the lengths. If, on the other hand, the principal errors are of the systematic class, the accumulated errors are proportional to the length, and the weights to be assigned are inversely as the squares of the length. The decision as to the assignment of weights must be based on the computer's judgment as to whether the lines should be assigned to one or the other of these classes, his judgment being based upon special investigation, if possible.

194. If the leveling combined in a net is done with various instruments and methods and under strongly contrasting conditions, the difficulty of assigning proper weights is largely increased. In such a case the basis for judgment as to the relative weights which should be assigned will be frequently found to be rather insecure. If the net of levels is extensive, and if there are several lines in each of the classes of leveling, the following principle may be used to determine whether the weights assigned are approximately correct and what modifications, if any, are desirable. As the correct weights are inversely proportional to the squares of the probable errors, it follows that in general, unless there is an extraordinary difference as to the manner in which the different classes of lines are involved in the net, that the average value of pv^2 for each class of lines should be the same, if the assigned weights are correct. If a certain class of lines be assigned too great weight in a given adjustment, the average pv^2 for that class will be larger than for other classes, and *vice versa*. New relative weights may be assigned as indicated by the average values of pv^2 for the different classes, and a new adjustment made which will, in turn, furnish a new test of the assigned weights. In making the changes in weights, it should be kept in mind that as the weights, p, for a given class

are decreased, there will be a tendency for the v^2s in that class to increase. It is not important to make a very close approximation to the ideal weights, for the reason that it will be found in general that a considerable change in weights is accompanied by only a small change in the numerical results of the adjustment. For an example of the application of this method of testing assigned weights in connection with the adjustment of the precise level net of the United States, see Appendix 8 of the Coast and Geodetic Survey Report for 1899, pp. 437, 438, 447, 448.

The same principle may be applied to testing the assigned relation between the lengths of the lines and their weights. If the values of pv^2 resulting from an adjustment for the various lines of a single class of leveling are tabulated in order of the lengths of the lines, the values should show no progressive change if the assigned relation between length and weight is correct. If, for example, on the other hand, weights inversely proportional to the length have been assigned to a given class of leveling, whereas they should have been inversely proportional to the square of the length, it will be found that the average pv^2 for the long lines is much greater than for the short lines. For an example of the application of the principle, see pp. 445–447 of the Appendix referred to in the preceding section.

This method of determining the proper weights becomes more reliable the more extensive the net and the greater the number of lines in each class.

195. It sometimes happens, as, for example, in the adjustment of the precise level net covering the eastern half of the United States, that the elevations are all referred to mean sea level by tidal observations connected directly with the net at various points. In this case the method of adjustment is the same as outlined above, sea level at *any* point being treated as if it were the one bench-mark in the above statement to which all levels are referred.*

* For examples of complicated level net adjustments, see Appendix 8, Coast and Geodetic Survey Report for 1899, and Appendix 3 of 1903.

196. The adjustment of a net of trigonometric levels does not differ essentially from that outlined above. In this case, from measurements of vertical angles at various triangulation stations, combined with the horizontal distances between these stations as determined by the triangulation, the differences of elevations between the inter-visible stations taken in pairs, are computed. These computed differences of elevation are treated as the direct results of the observation, and the relative elevations of the various stations are derived from an adjustment of the net.

CHAPTER IX

APPLICATION TO THE SELECTION OF METHODS OF OBSERVATION

197. The preceding chapters have dealt with the principles of least squares as furnishing a method of computing the most probable results from given observations. To confine the attention entirely to this application, as is frequently done, even by persons thoroughly familiar with the principles, is to overlook a still more important though less extensive use to which these principles may be applied, namely, to the selection of methods of observation.

When, in any class of precise measurement, the method of observation is selected, the maximum accuracy attainable is thereby fixed. The computer cannot improve upon what the observer has done, he can but bring out all its excellence. The observer may, by improvements in his method of observation, not only raise the standard of maximum accuracy attainable in his own observations, but by example raise it for all later observers. Or he may for himself and later observers greatly reduce the amount of observing necessary to attain a given standard of accuracy, and thereby greatly reduce the cost of the work. Improvements in methods of observation are far-reaching in their effects, and the application of least squares to this end correspondingly important.

The expression, "selection of methods of observation," is here used in a general sense such as to include the selection of the instrument and its mounting and protection, and the selection of conditions under which to observe, as well as the mere selection of a method of manipulation and a program of observation.

198. A clear understanding of the relative influence upon

the results of errors arising from various sources is a prime requisite on the part of one who proposes to improve methods of observation. It is also of great importance to the observer attempting to secure a maximum of accuracy with a minimum of expenditure. A working knowledge of the principles of least squares is essential to this clear understanding.

In general, methods of observation are to be improved:

1. By reducing at their source the errors which have predominating influence.

2. By transferring the errors from any given source from the systematic or constant class into the accidental class, by a change in instrument or method.

3. By introducing such simplifications in instruments and methods as will increase the rapidity of observing, possibly at the expense of making slight increases in such errors as have little influence on the final results. In general, the third suggestion is important not simply for economic reasons, but also because any increase in the rapidity of observing is likely to lead indirectly to an increase in accuracy by reducing instrumental errors.

A treatment of the application of the principles of least squares to the selection of methods of observation within the limits of a single chapter must be suggestive rather than complete. In the examples given to illustrate this application, considerable knowledge of instrument and method of observation will be assumed to be possessed on the part of the reader. Attention is invited to the principles sketched rather than to the particular numerical estimates of magnitude of each class of errors in the examples.

199. Distinction Between Accidental, Systematic, and Constant Errors. — In discussing errors, and especially when discussing them with reference to their ultimate effects, it is quite important to keep clearly in mind the distinctions between accidental errors, constant errors, and systematic errors. A *constant error* is one which has the same effect upon all the

observations of the series or portion of a series under consideration. A *systematic error* is one of which the algebraic sign, and, to a certain extent, the magnitude, bears a fixed relation to some condition or set of conditions. *Accidental errors* are not constant from observation to observation, they are as apt to be minus as plus, and they presumably follow the law of error which is the basis of the theory of least squares. Thus, for example, the phase error in observations of horizontal directions is systematic with respect to the azimuth of the sun and of the line of sight. The personal equation of an observer introduces a constant error into the observations of the separate stars in a time set. The expression "constant error" is often used loosely in contradistinction to "accidental error," in such a way as to include both strictly constant errors and systematic errors.

The effect of accidental errors upon the final result may be diminished by continued repetition of the observations and by the least-square method of computation. The effects of constant errors and of systematic errors must be eliminated by other processes, for example, by changing the method or program of observations, by special investigations or by special observations designed to evaluate a constant error or to determine the exact law of a systematic error.

200. More Accurate Definition of Probable Error. — It cannot be emphasized too much or too frequently that the theory of least squares applies to accidental errors only, that the least-square method of computation is designed to secure efficient elimination of the effects of accidental errors, but may or may not be efficient in reducing the effects of errors of other kinds, and that a probable error derived directly from residuals is an adequate measure of accidental errors only. The most frequent, and perhaps the most serious, mistakes made in applying the theory of least squares arise from disregarding the points here emphasized.

The definition of probable error as ordinarily given is equiva-

lent to saying that the chances are even for or against the proposition that a certain stated value having given a probable error does not differ from the truth by more than that probable error. That is, if the probable error of a single observation of an angle is stated to be $\pm$ 1.00″ it is understood that the chances are even for and against the proposition that the result of any one observation is within 1.00″ of the truth.

If the probable error in question is one which is based directly upon residuals, this definition should be modified to limit it so as to refer to the accidental errors only. This may be done by stating that in the above case the chances are even for or against the proposition that the result of any one observation is within 1.00″ of the mean value which would result from an infinite number of such observations made under the same average conditions as the observation in question. This form of definition is non-committal as to possible constant errors affecting all the observations, and as to systematic errors which might change in magnitude and algebraic sign, if there were changes in the conditions. Such a definition differs from the ordinary definition in stating that the probable error is a measure of the departure to be expected from a mean of an infinite number of such results as are being considered, rather than from the truth. The supposed mean of an infinite number of results would be free from the effects of accidental error, would be as much affected by constant error as any one observation, and would be less subject to systematic error only to the degree determined by the frequency and extent to which the conditions were allowed to vary.

There is no sharply defined line of separation between accidental and systematic errors. The perfect type of accidental error, the type upon which the theory of least squares is based, is an error* which is the algebraic sum of an infinite number of independent infinitesimal elemental errors, all equal in magnitude, and each as likely in any given case to be positive as

* See p. 161.

negative. In any actual case the number of elemental errors is finite, each of them is finite in magnitude, and the elemental errors due to different causes may be of very different average magnitude. Each elemental error actually depends both in magnitude and sign upon certain conditions, though those conditions may be unknown, and is, therefore, a systematic error. The conditions upon which different elemental errors depend may not be independent. Indirect evidence shows nevertheless that in many cases the errors of observation were sensibly of the perfectly accidental type, that is, the relation between the numbers of errors of various magnitudes and signs is sensibly that which must hold for the perfect type of accidental error (see p. 31 and Table I), and therefore all deductions based upon this law of error are sensibly true. In other contrasting cases, certain of the elemental errors may be easily proved to be in the systematic class, and may be segregated from the remaining elemental errors which then, as combined, belong sensibly to the accidental class. In the intermediate and most frequent cases, there are slight observable departures from the law of error corresponding to the perfect type of accidental error, and the deductions based upon that law are slightly, but observably, at variance with the facts. In such a case some systematic error exists which is almost, but not quite, detected, and one is forced to treat all the errors as being of the accidental class.

201. Detecting Systematic or Consatnt Errors. — It is important that the computer should detect systematic or constant errors in order that he may modify his method of computation to correspond. It is still more important that the investigator of methods of observation should detect systematic and constant errors in order that they may thereafter be avoided. It is much more important to avoid, or to provide special means of eliminating, systematic or constant errors than accidental errors, for the reason that accidental errors are rapidly eliminated by the mere process of increasing the number of observations.

A considerable variety of methods may be used for the detection of systematic and constant errors. Each method will be found as a rule to correspond to one or another of the following five cases:

CASE 1. Systematic errors may sometimes be detected by noting a tendency of the residuals to have a certain algebraic sign when certain conditions exist, and the opposite sign when these conditions are absent or the opposite conditions occur.

CASE 2. Errors which are constant in each of a number of groups of observations may sometimes be detected by noting that the disagreements between the mean results for each group are greater than can be accounted for by the probable errors of those means as computed from the residuals within each group. Or, what is equivalent, errors which are constant for each of a number of groups of observations may sometimes be detected by noting that the probable error of the mean of all the observations is apparently much larger if computed from the residuals of the mean of each group from the mean of all, than if computed from the residuals of each observation from the mean of all. When it is recognized that there are errors which are constant for each group, these errors may in some cases be proved to belong to the systematic class by detecting a relation between the errors peculiar to each group and some condition peculiar to that group.

CASE 3. Systematic errors may sometimes be detected by comparing the relative frequency of residuals of different magnitude and sign with the theoretical relation, according to the law of error, between the magnitude and the frequency of errors. This comparison may be made by using Table I, or by plotting the actual curve of the residuals with the theoretical curve of error superposed upon it. The comparison may show that there are relatively many more very large residuals than would be the case if the errors were all accidental. The examination of the conditions corresponding to these large residuals may then lead to the detection of the law and cause of these large systematic errors.

Case 4. That systematic or constant errors may arise from a given source may sometimes be proved by special observations for that purpose.

Case 5. Either systematic or constant errors may sometimes be detected by comparing the results of the observations in question with results obtained independently from observations of an entirely different kind.

The detection of systematic or constant errors necessarily involves least squares as a basis, but this must be supplemented by something else, as the method of least squares deals with accidental errors only.

Examples of each of these cases will be found in the text which follows:

202. Zenith Telescope Latitude Observations. — Observations with a zenith telescope for latitude are especially interesting as illustrating errors which are a close approximation to the perfect type of accidental error. The zenith telescope and instrument is an example of proper selection. It is difficult to improve upon it for the reason that the errors from every important source are of the accidental class, and the effects of the errors arising from various sources upon the final results are so nearly of the same magnitude that little gain in accuracy may be secured except by reducing the errors from several sources.

As indicating how the error in a result from an observation of a single pair of stars with a zenith telescope approaches the ideal accidental error, which is supposed to be the algebraic sum of an infinite number of independent infinitesimal errors, all equal in magnitude, and each as likely in any case to be positive as negative, sixteen independent elemental errors in this result may be named, each capable of introducing an accidental error of from $\pm 0.01''$ to $\pm 0.16''$ into the result, of which the probable error due to all causes is $\pm 0.20''$ to $\pm 0.30''$. Several of the sixteen errors named could if desired be separated by more minute examination into other elemental errors as suggested in the text, so that the number of elemental errors is

really several times sixteen. The list of elemental errors as given is suggestive rather than complete.

203. Elemental Errors in Zenith Telescope Observations. — 1. The error of bisection of the star image by the micrometer line, depending among other things upon the observer's perception and his control of the muscles of his fingers, the shape of the image, the defects of the observer's eyes, the irregular motion of the image due to momentary changes in refraction, irregularities in the line used in making the bisection, the magnitude of the star and the lighting of the field of view and the line.

2. The error of reading the position of the bubble in the level, depending upon the lighting and upon parallax as well as upon the observer's estimate of tenths of a division and his perception.

3. The error caused by unequal heating of the level vial and consequent movements of the bubble.

4. The error caused by the error in the assumed value of one division of the level. The error in the value of a division is due in part to errors in determining it and in part to variation of the actual value from time to time.

5. The error of reading the micrometer head.

6. The error due to the error in the assumed value of one turn of the screw. The remark made in connection with the level value applies here also.

7. The error due to non-uniformity of the screw throughout its length.

8. The error due to periodical errors, having a period of one turn in the screw and its nut.

9. The error due to the inclination of the bisecting line to the horizon and to the difficulty of making all bisections on exactly the same part of the line.

10. The error due to inclination of the horizontal axis of the instrument.

11. The error due to the azimuth error of the instrument.

12. The error due to the collimation error of the instrument.

13. The error due to variations in the angle between the tangent to the level vial at its middle point and the line of collimation, this variation being due to changes in temperature in different parts of the instrument as well as to stresses.

14. The error due to variation of the differential refraction from its assumed mean value. This variation is dependent upon the conditions as to temperature and pressure at all points along many miles of each of the two lines of sight.

15. The error due to the error in the declination of each star as used in the computation. The error of declination is due in general to the separate errors in dozens or perhaps even hundreds of observations made at various times at many different observatories, each observation being in general affected by as many elemental errors as are suggested in the preceding fifteen numbers of this list.

In estimating the number of elemental errors affecting a result which are represented by this list, it should be noted that each result depends upon the observations on two stars.

204. In determining the latitude of a station by zenith telescope method, a program frequently followed is to observe several pairs of stars, say twenty, on each of several nights, five, for example. The probable error of a single observation is computed from the residuals of each observation from the mean of the five observations on that pair. When the mean results for the different pairs are compared, they are found to show disagreements which are greater than can be accounted for by the residuals within each group. This test, which is an illustration of Case 2, shows that there is some error in the results which is constant for each group. In this case it seems obvious that this constant error in each group is, in the main, simply the error in the mean of the two declinations of the stars of the pair. This declination error, which is constant for each pair, belongs mainly in the accidental class when the results from various pairs are considered.

There may be other errors which are constant for each pair and which combine with these declination errors; for example, errors No. 6 and 7, and some parts of No. 1 of the above list. For the present purpose it is not sufficient to stop with the obvious conclusion that all or nearly all of the error which is constant for each pair is due to declinations. It is desirable, if possible, to prove it. Two lines of proof are available and have been used.

Let e_p be the probable error of the mean result from a single pair as derived from the residuals of these various mean results from the mean of all for the station. Let ϵ be the average probable error of a mean result for a pair as derived from the residuals of the separate observations on that pair from the mean result for that pair. Let $e_{\frac{**}{2}}$ be the average probable error of the mean of two star declinations for the particular stars observed. The apparently obvious assumption suggested in the preceding two paragraphs is expressed by the formula:*

$$e_p^2 = \epsilon^2 + e^2_{\frac{**}{2}}.$$

$e_{\frac{**}{2}}$ is the only unknown, and its value may therefore be computed from the latitude observations. Its value may also be derived from the computations made by the astronomer in combining the observations at various observatories and at various times. These two computations give usually nearly the same values for $e_{\frac{**}{2}}$, the value from the latitude observations being, upon the average, very slightly larger than that computed by the astronomer. Hence, the errors which are constant for each pair are nearly, but not quite, all due to errors of declinations.

If several latitude stations along the same parallel have been occupied and the same list of pairs used at all stations, it becomes possible to apply in a slightly different way the principles involved in the preceding paragraph. This was done for twelve

* See Appendix 7 of the Coast and Geodetic Survey Report for 1898, Longitude, Latitude, Azimuth, p. 358.

stations along the Mexican boundary. It confirmed the conclusion of the preceding paragraph.*

205. Let it be supposed that the probable error of the mean of the two declinations, $e_{\frac{**}{2}}$, is $\pm 0.16''$, and that the probable error of a single observation of a pair is $\pm 0.30''$. The first of these probable errors represents the accuracy to be expected in the declinations which are now available. The second is easily attained with a good portable zenith telescope. Under these conditions, which is the better program, to observe 20 pairs on each of five nights, 100 observations in all, five on each pair, or to observe 100 pairs each once only, the observations being scattered over as many nights as may be necessary to secure them? If 20 pairs are observed on each of five nights, the probable error of the final result will be,

$$\sqrt{\frac{e^2_{\frac{**}{2}}}{20} + \frac{e^2}{100}} = \sqrt{(.036)^2 + (.030)^2} = \pm .047.$$

If 100 pairs are each observed once, the probable error of the final result will be

$$\sqrt{\frac{e^2_{\frac{**}{2}}}{100} + \frac{e^2}{100}} = \sqrt{(.016)^2 + (.030)^2} = \pm .034,$$

showing a very decided advantage in favor of this program of observation. The advantage becomes still more evident when it is noted that if 54 pairs be each observed once, the probable error of the final result will be

$$\sqrt{\frac{e^2_{\frac{**}{2}}}{54} + \frac{e^2}{54}} = \sqrt{(.022)^2 + (.041)^2} = \pm .047,$$

the same as would be obtained from 100 observations on 20 pairs, but secured with little more than one-half as much observing. It would evidently be a great improvement to change

* Report of the Boundary Commission upon the Survey and Re-marking of the Boundary between the United States and Mexico, West of the Rio Grande, 1891–1896, p. 105.

from the plan usually followed in the place of observing each pair four or more times to the plan of observing each pair but once. This improvement is one which almost inevitably suggests itself to a person looking at the subject from a least-square point of view, whereas the writer's experience indicates that those who do not take this view-point fail to appreciate the desirability of this improvement. For an example of the comparative results by the two plans, see the Mexican Boundary Report referred to on the preceding page, pp. 106, 107. At the typical station represented by that series of observations, 108 observations on 72 pairs gave a much more accurate result than could have been obtained even from an infinite number of observations on 18 pairs.

206. Telegraphic Longitude Observations.—Determinations of differences of longitude by the telegraphic method furnish illustrations of the detection of systematic or constant errors by the methods of Case 1 and Case 4 and the following three illustrations of Case 2.

1. For the transits ordinarily used in telegraphic longitude determinations in the Coast and Geodetic Survey, the probable error of an observing transit of a star over a single line is usually less than $\pm 0.10''$, as computed from the residuals of the separate lines from the mean of the 11 lines on which observations were taken.* On this basis, the probable error of the transit across the mean of the 11 lines would be less than $\pm 0.03''$. This same probable error, as computed from the residuals of the separate stars from the mean for the time set, is usually considerably larger, say $\pm 0.04''$ upon an average. This indicates that the errors of the observations would not be much reduced by increasing the number of lines, say from 11 to 21.

2. From fifteen Coast and Geodetic Survey longitude determi-

* The numerical estimates of errors used in this chapter are taken, as a rule, from Appendix 7 of the Coast and Geodetic Survey Report for 1898, Time, Longitude, Latitude, and Azimuth.

nations involved in the primary longitude net, it was found that whereas the probable error of a difference of longitude from one night's observation as computed from the residuals of different stars from the mean for the night was $\pm 0.013''$, if this probable error were computed from the residuals of different nights from the mean of the ten nights concerned in the determination (the correction for relative personal equation being already applied), it was enough larger to indicate that the constant error peculiar to each night was ± 0.022. Hence there would be little appreciable gain in accuracy if the number of observations per night were greatly increased.

3. From many longitude determinations involved in the primary longitude net, and each consisting as a rule of ten nights of observation, it was found that whereas the probable error of the mean of the ten nights as computed from the residuals of the separate nights from the mean was $\pm 0.011''$, the probable error as computed from the adjustment of the net was so much larger as to show that there was a constant error peculiar to each mean of ten nights of $\pm 0.022''$. It follows that a reduction in the number of nights to six or even four would result in but a slight increase in accuracy, — say 10 per cent.

In the longitude determinations referred to above, the usual procedure was for the two observers to exchange places after the first five of the ten nights of observation. In all other respects except this, the last five nights of observation were made under conditions as nearly as possible identical with those during the first five nights. If the mean of the ten nights is taken, and the corresponding residuals written out, it is evident, as a rule, that there is a tendency for the residuals of one group of five to have one sign, and of the other group to have the opposite sign. This is an illustration of Case 1, and indicates that there is a systematic error in the result which bears a fixed relation to the relative position of the observers, and is therefore due to their relative personal equation. Accordingly it is eliminated by taking one-half the difference of the two groups

of five as the relative personal equation, correcting each night's result by this amount, and deriving the final difference of longitude from these corrected values. A confirmation of the supposition that this systematic error is the relative personal equation, is furnished by comparing successive values of the relative personal equation as thus derived for the same pair of observers in successive longitude determinations.

In connection with some longitude determinations, the relative personal equation of the two observers has been determined by a personal equation machine, or by observations by the half-transit method. Either of these is an illustration of Case 4. Similarly the method of Case 4 has been applied to prove that the systematic error in a longitude determination arising from the action of the single relays which connect the mean telegraphic line at each longitude station with the chronograph circuit must be in the thousandths of seconds only, not in hundredths. This was done by making special observations of changes which occur in the time of operation of these relays when extreme changes are made in their adjustment and in the strength of the currents operating them.

A study of the sources of error in telegraphic longitude determinations by the methods suggested in this chapter, but necessarily in much greater detail than it is possible to give here, inevitably leads to the conclusion that a considerable portion of errors which are constant for a night are due to variation of the relative personal equation of the two observers. This suggests that the line along which some improvements in the methods should be sought is that of securing some means of making the relative personal equation and its variation zero. For this purpose a new attachment to the astronomical transit, known as a transit micrometer, was put into use several years ago by the Prussian Geodetic Institute with great success, and is now in use in this country.*

* See Appendix 8 of the Coast and Geodetic Survey Report for 1904, The Transit Micrometer.

207. Other Illustrations. — As another illustration of Case 2, it may be noted that in the measurement of angles in primary triangulation the general experience is that the probable error of an angle as computed from the residuals of the various measures of that angle from their mean, or even computed from the residuals which occur in a local adjustment involving all the angles at a station, is very much smaller than the probable error of an angle computed from the figure adjustment, which involves a much larger group of observations. This indicates that there are errors which are constant for a station and are not affected by either increasing or decreasing the number of measurements of an angle. Acting upon this reasoning, the Coast and Geodetic Survey has recently reduced the number of observations of each angle in primary triangulation from 22 to 34, to 16. This very important saving in time and money has not been accompanied by any appreciable decrease in accuracy.

208. An examination of observations of astronomical azimuth at many stations has shown that frequently the residuals for each night tend to stand in a group by themselves, all having one sign. This examination is one form of application of the method of Case 2. It indicates that there are constant errors, in some instances, at least, peculiar to each night in azimuth observations, and that therefore, if the highest degree of accuracy is desired, the observations must be extended over several nights. The same test applied to latitude observations made with a zenith telescope, indicates in general that there are no constant errors peculiar to each night, though to this statement some exceptions have been noted.

209. Two interesting illustrations of Case 2 are furnished by the use of a five-meter iced bar in the standardization of base apparatus in the Coast and Geodetic Survey. In determining the length of the bar, it was found that the residuals apparently indicated that the bar was 2 to 4 microns longer when its length was determined while its A-end lay to the left of the observer

than when it lay in the reverse position.* Though this is a very small quantity, it was persistently shown by the observations. It was recognized, when it had been carefully studied, as a systematic error due to the personal equation of the observers in making bisections of the graduations on the bar. To secure the highest possible degree of accuracy in determining the length of the bar and in using it as a standard, it is therefore necessary to make one-half the observations with the bar in each position and therefore eliminate this systematic error.

Again, in using this iced bar in the open air to measure a standard length of 100 meters, it was found that the residuals from the mean of all the measures was of one sign if the measurement had progressed toward the sun, and of the opposite sign if the measurement had progressed away from the sun.† This systematic error was eliminated, in part at least, by making the measurements in pairs with the progress in opposite directions in the two measurements of each pair, and with as short an intervening time interval as possible. This systematic error is believed to be due to slight motions of the microscopes due to changes of temperature.

210. A good illustration of Case 3 is furnished by a comparison of observed and predicted tides at Sandy Hook, N.J.‡ The difference between the observed and predicted heights of high and low water was due to many separate elemental errors, the errors in the six years of tidal observations from which the tidal constants used in the prediction were derived and in the one year of observations with which the comparison was made, the errors in the theory involved in the computation of the tidal constants, and the errors in the operation of the predicting machine itself. It was desired to determine as fully as pos-

* Appendix 8, Coast and Geodetic Survey Report for 1892, the Holton Base, pp. 382–391.

† Appendix 3, Coast and Geodetic Survey Report, 1901, the Measurement of Nine Bases, p. 245.

‡ Appendix 15, Coast and Geodetic Survey Report for 1890, and especially illustrations Nos. 66 and 67 of that Report.

sible what systematic errors existed, the magnitude of the accidental errors, and especially how large were the errors due to the machine.

There were four groups of 700 each to be considered, arising respectively from the prediction of high-water heights, low-water heights, high-water times and low-water times. In addition to other tests applied, the probable error of a single prediction was computed for each of these groups (after the constant error had been removed), the curve of distribution of errors for each group drawn on a large scale (the magnitudes of the error being the abscissæ, and the number of such errors the ordinates of the curve), and the theoretical law of error (Art. 27) drawn to the same scale was superposed on it.

For the two curves corresponding to predicted heights it was at once apparent that there was a systematic difference in character between the actual and theoretical curves. The actual curve in each case showed about 13 times as many errors as the theoretical curve greater than $4\frac{1}{2}$ times the computed probable error, and about 79 times as many greater than $5\frac{3}{4}$ times the computed error. It also showed about $\frac{7}{8}$ as many errors as the theoretical curve less than the computed probable error, and about $\frac{6}{7}$ as many between one and three times the computed probable error. A difference of this kind between the actual and theoretical curves indicates that there is, in addition to the actual errors, some large systematic error which occurs occasionally, in this case about one in twenty times upon an average. The unusual number of large residuals would be caused directly by such a systematic error. An indirect effect of these large residuals, due mainly to the large systematic error superposed on the accidental errors, would be to make the computed probable error much too large. This in turn would cause the theoretical curve to depart from the actual between errors of zero and three times the probable error, in a manner similar to that noted above. Following up the conclusion that the particularly large residuals indicated by the outer portions of

the actual curve were due to a systematic error, considerable evidence was found that they were due to effects of storms upon mean sea level. As the principal purpose of the test was to determine how great were the errors due to the action of the tide predicting machine, this conclusion that the large errors were not chargeable to the machine was an important one.

The comparison between the actual and theoretical curves for predicted times of high and low water showed a very close agreement. Storms are known to have but little effect upon the time of high and low water, hence the systematic errors due to this cause should be expected to be small, as the curves indicated them to be.

211. Trigonometrical leveling, that is, leveling by observations of vertical angles taken in connection with triangulation, frequently connects points that are also connected by precise leveling, and thus furnish an illustration of Case 5. The test applied by the precise level usually, but not always, indicates that the systematic and constant errors in theoretical leveling are so small as to be almost or quite concealed by the accidental errors.

212. The following three illustrations of Case 5 are all taken from *The Solar Parallax and Its Related Constants*, by William Harkness.

1. The aberration constant has been determined many times and by many methods, among which are: *a*, by observations of right ascensions of stars with an instrument in the meridian; *b*, by observations of the declinations of stars with an instrument in the meridian; *c*, by observations with an instrument in the prime vertical; *d*, by zenith telescope observations. A comparison of the results by the various methods indicates clearly that they are, as a rule, subject to constant or systematic errors much larger than the uneliminated effects of accidental errors.

2. The flattening of the earth has been derived: *a*, from geodetic arcs; *b*, from pendulum observations; *c*, from the observed precession and nutation; *d*, from perturbations of the

moon. The comparison of the various results indicates that the systematic or constant errors in some if not all of them are much larger than the uneliminated effects of accidental errors.

3. The mean density of the earth has been determined: *a*, by observations of the attraction of mountains as measured by the deviations of the plumb-line in the immediate vicinity; *b*, by observing the attraction of known masses of matter either with a torsion balance or a pendulum; *c*, by pendulum observations at different distances from the center of the earth, near mean sea level and on mountain tops, or at the surface of the earth and in mines; *d*, by observations with balances of the ordinary form either of the attraction of a known mass or of the change in the attraction of the earth upon a known mass when it is moved to a higher or a lower position. The comparison shows systematic or constant errors in the results which are large in comparison with the uneliminated accidental errors, as a rule, when methods *a* and *c* are used, and in some cases even when other methods are used.

APPENDIX

213. Values of $\Theta(t) = \frac{2}{\sqrt{\pi}} \int_0^{\rho \frac{a}{r}} e^{-t^2}\,dt.$

See Art. 22.

$\Theta(t)$ is the probability that the error will be less numerically than the limit which is expressed in the first column in terms of the p. e. Thus, the sixth line of the table means that out of every thousand errors the chances are that 264 will be less than one-half as great as the p. e.

The probability that an error a is greater than r is 0.5, than $2r$ is 0.177, than $3r$ is 0.043, than $4r$ is 0.007, than $5r$ is 0.001, than $6r$ is 0.0001.

TABLE I.

See Art 23.

$\frac{a}{r}$	$\Theta(t)$	DIFFERENCE.	$\frac{a}{r}$	$\Theta(t)$	DIFFERENCE.
0.0	0.000	..	2.5	0.908	..
0.1	0.054	54	2.6	0.921	13
0.2	0.107	53	2.7	0.931	10
0.3	0.160	53	2.8	0.941	10
0.4	0.213	53	2.9	0.950	9
0.5	0.264	51	3.0	0.957	7
0.6	0.314	50	3.1	0.963	6
0.7	0.363	49	3.2	0.969	6
0.8	0.411	48	3.3	0.974	5
0.9	0.456	45	3.4	0.978	4
1.0	0.500	44	3.5	0.982	4
1.1	0.542	42	3.6	0.985	3
1.2	0.582	40	3.7	0.987	2
1.3	0.619	37	3.8	0.990	3
1.4	0.655	36	3.9	0.991	1
1.5	0.688	33	4.0	0.993	2
1.6	0.719	31	4.1	0.994	1
1.7	0.748	29	4.2	0.995	1
1.8	0.775	27	4.3	0.996	1
1.9	0.800	25	4.4	0.997	1
2.0	0.823	23	4.5	0.998	1
2.1	0.843	20	4.6	0.998	0
2.2	0.862	19	4.7	0.998	0
2.3	0.879	17	4.8	0.999	1
2.4	0.895	16	4.9	0.999	0
2.5	0.908	13	5.0	0.999	0

TABLE II.

214. *Factors for Bessel's Probable Error Formulas.*

See Art. 33.

n	$\frac{.6745}{\sqrt{n-1}}$	$\frac{.6745}{\sqrt{n(n-1)}}$	n	$\frac{.6745}{\sqrt{n-1}}$	$\frac{.6745}{\sqrt{n(n-1)}}$
..	...	...	40	0.1080	0.0171
..	...	...	41	.1066	0.0167
2	0.6745	0.4769	42	.1053	0.0163
3	.4769	.2754	43	.1041	0.0159
4	.3894	.1947	44	.1029	0.0155
5	0.3372	0.1508	45	0.1017	0.0152
6	.3016	.1231	46	.1005	.0148
7	.2754	.1041	47	.0994	.0145
8	.2549	.0901	48	.0984	.0142
9	.2385	.0795	49	.0974	.0139
10	0.2248	0.0711	50	0.0964	0.0136
11	.2133	.0643	51	.0954	.0134
12	.2029	.0587	52	.0944	.0131
13	.1947	.0540	53	.0935	.0128
14	.1871	.0500	54	.0926	.0126
15	0.1803	0.0465	55	0.0918	0.0124
16	.1742	.0435	56	.0909	.0122
17	.1686	.0409	57	.0901	.0119
18	.1636	.0386	58	.0893	.0117
19	.1590	.0365	59	.0886	.0115
20	0.1547	0.0346	60	0.0878	0.0113
21	.1508	.0329	61	.0871	.0111
22	.1472	.0314	62	.0864	.0110
23	.1438	.0300	63	.0857	.0108
24	.1406	.0287	64	.0850	.0106
25	0.1377	0.0275	65	0.0843	0.0105
26	.1349	.0265	66	.0837	.0103
27	.1323	.0255	67	.0830	.0101
28	.1298	.0245	68	.0824	.0100
29	.1275	.0237	69	.0818	.0098
30	0.1252	0.0229	70	0.0812	0.0097
31	.1231	.0221	71	.0806	.0096
32	.1211	.0214	72	.0800	.0094
33	.1192	.0208	73	.0795	.0093
34	.1174	.0201	74	.0789	.0092
35	0.1157	0.0196	75	0.0784	0.0091
36	.1140	.0190	80	.0759	.0085
37	.1124	.0185	85	.0736	.0080
38	.1109	.0180	90	.0713	.0075
39	.1094	.0175	100	.0678	.0068

TABLE III.

215. *Factors for Peters' Probable Error Formulas.*

See Art. 33.

n	$\frac{.8453}{\sqrt{n(n-1)}}$	$\frac{.8453}{n\sqrt{n-1}}$	n	$\frac{.8453}{\sqrt{n(n-1)}}$	$\frac{.8453}{n\sqrt{n-1}}$
..	...	...	40	0.0214	0.0034
..	...	...	41	.0209	.0033
2	0.5978	0.4227	42	.0204	.0031
3	.3451	.1993	43	.0199	.0030
4	.2440	.1220	44	.0194	.0029
5	0.1890	0.0845	45	0.0190	0.0028
6	.1543	.0630	46	.0186	.0027
7	.1304	.0493	47	.0182	.0027
8	.1130	.0399	48	.0178	.0026
9	.0996	.0332	49	.0174	.0025
10	0.0891	0.0282	50	0.0171	0.0024
11	.0806	.0243	51	.0167	.0023
12	.0736	.0212	52	.0164	.0023
13	.0677	.0188	53	.0161	.0022
14	.0627	.0167	54	.0158	.0022
15	0.0583	0.0151	55	0.0155	0.0021
16	.0546	.0136	56	.0152	.0020
17	.0513	.0124	57	.0150	.0020
18	.0483	.0114	58	.0147	.0019
19	.0457	.0105	59	.0145	.0019
20	0.0434	0.0097	60	0.0142	0.0018
21	.0412	.0090	61	.0140	.0018
22	.0393	.0084	62	.0137	.0017
23	.0376	.0078	63	.0135	.0017
24	.0360	.0073	64	.0133	.0017
25	0.0345	0.0069	65	0.0131	0.0016
26	.0332	.0065	66	.0129	.0016
27	.0319	.0061	67	.0127	.0016
28	.0307	.0058	68	.0125	.0015
29	.0297	.0055	69	.0123	.0015
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